Path-dependent implied volatility surfaces

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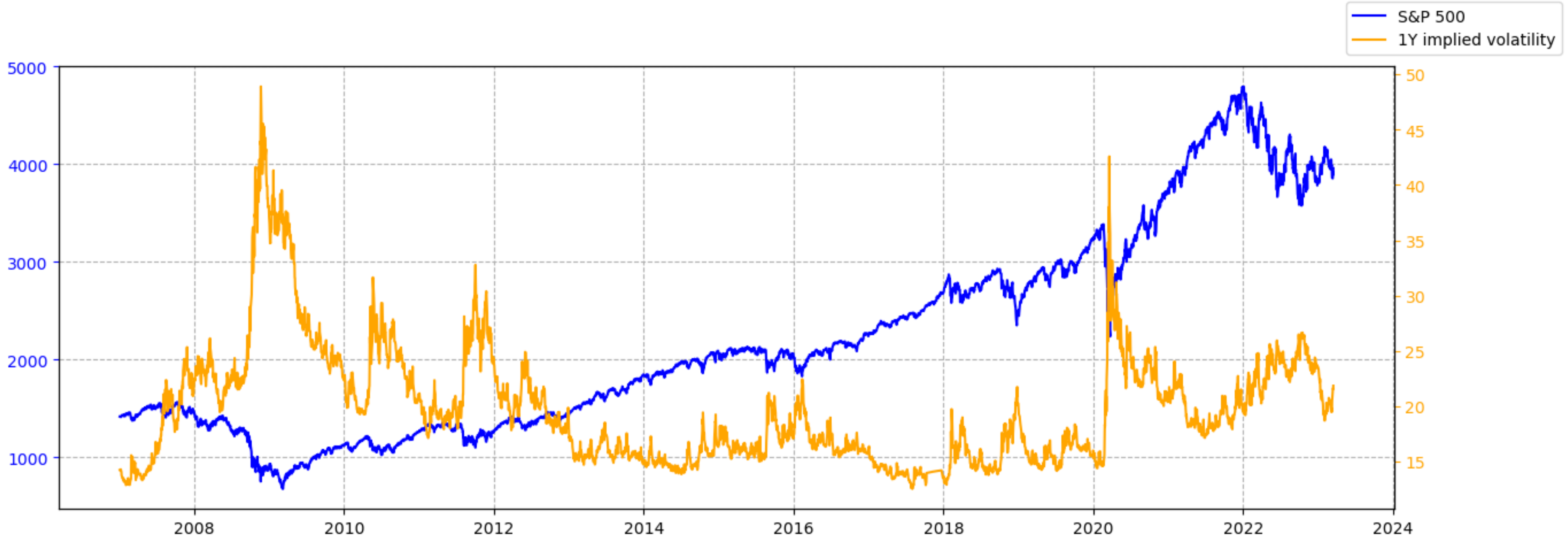
About the speaker

- **Hervé Andrès –** *R&D Consultant, Milliman Paris*
  → 3rd year PhD student in collaboration with the CERMICS research center (Ecole des Ponts, Paris)
  → Research topics: financial modelling and validation of real-world economic scenarios for applications in the insurance industry

- Milliman is a global actuarial consulting firm with expertise in Health, Insurance, Retirement & Benefits and Risk.

Introduction and motivation

- **Real-World Economic Scenario Generators (RW ESGs)** → key tools in the insurance industry for various applications: ALM studies, strategic asset allocation, regulatory capital requirements, etc.
- Within most of the current RW ESGs, **implied volatility** is not modelled or is modelled using simple models that **do not reproduce some of the more important stylized facts** of historical data such as **spikes** during market crashes.



Introduction and motivation

- **Reminder:** given the price $C(K,T)$ of an European call option with strike K and maturity T, there exists a unique parameter $\sigma > 0$, called the **implied volatility**, such that $C_{BS}(K,T,\sigma) = C(K,T)$ where C_{BS} is the Black-Scholes call option price.
- According to the Black-Scholes model : the implied volatility is the **same for all maturities and strikes** and is **constant over time**.
- The computation of the implied volatility from market option prices shows that :
  1. it actually depends both on the maturity and the strike $\rightarrow$ we actually have an **implied volatility surface (IVS):** $(K,T) \mapsto \sigma_{BS}(K,T)$;
  2. the shape and the level of the IVS varies with time.
- **Contribution:** stochastic model for the **joint evolution** of the IVS and its underlying asset price.
- **Key idea:** make the implied volatility evolve as a function of the **path** of the underlying asset price.

Guyon and Lekeufack's PDV model

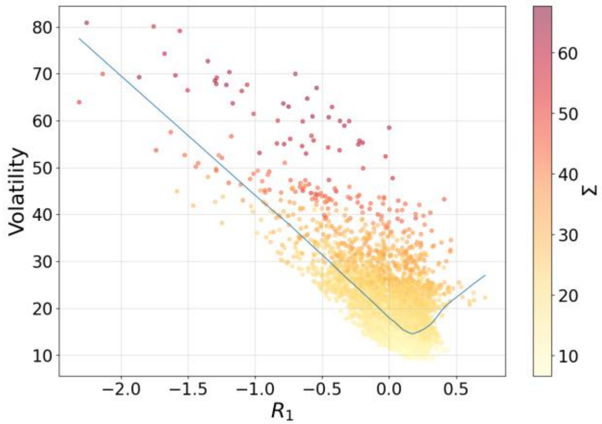
- Guyon and Lekeufack (2023) claim that **volatility is mostly path-dependent.**
- This claim is based on an empirical study relying on the following model (the **Path-Dependent volatility** model):

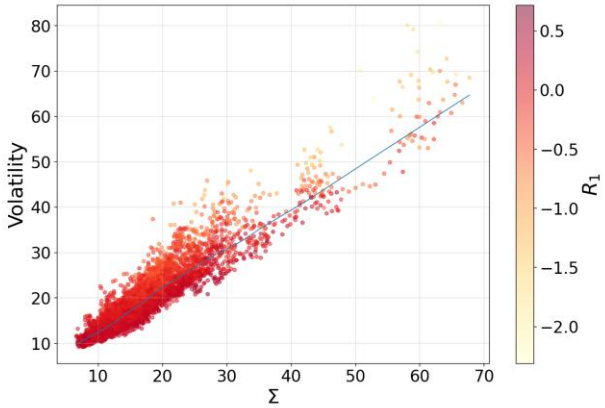
$$\boldsymbol{Volatility_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \Sigma_t}$$

  - ➢ $Volatility_t$ is either the value of a volatility index (e.g. the VIX) or a measure of realized volatility.
  - ➢ R_1 and Σ are respectively trend and volatility features:

$$R_{1,t} = \sum_{t_i \le t} \frac{Z_{\alpha_1,\delta_1}}{(t-t_i+\delta_1)^{\alpha_1}} r_{t_i} \text{ and } \Sigma_t = \sqrt{\sum_{t_i \le t} \frac{Z_{\alpha_2,\delta_2}}{(t-t_i+\delta_2)^{\alpha_2}} r_{t_i}^2}$$

with $r_{t_i} = \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}$ the daily returns and Z_{α_i,δ_i} a normalization constant.





- **Calibration on historical data:** R^2 scores above 87% on the train set and above 80% on the test set → the **PDV model explains a large part of the variability observed in the volatility dynamics**.

Agenda

1. Introduction and motivation
2. Empirical study
3. The path-dependent SSVI model
4. Application to RILA hedging
5. Key takeaways

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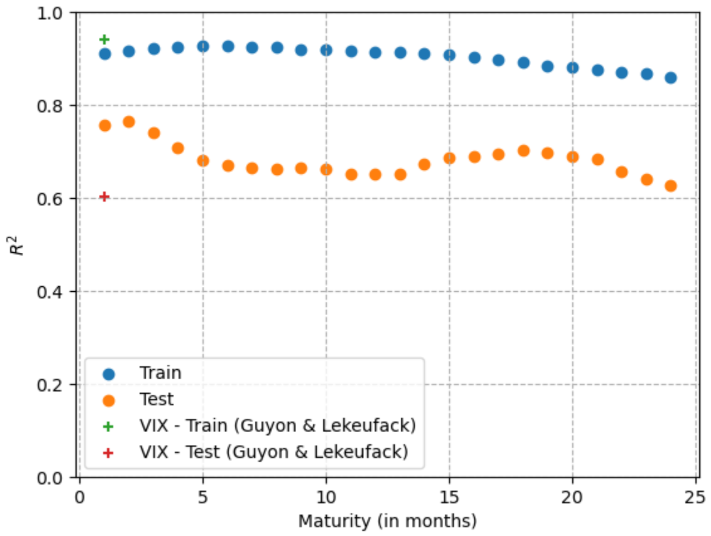
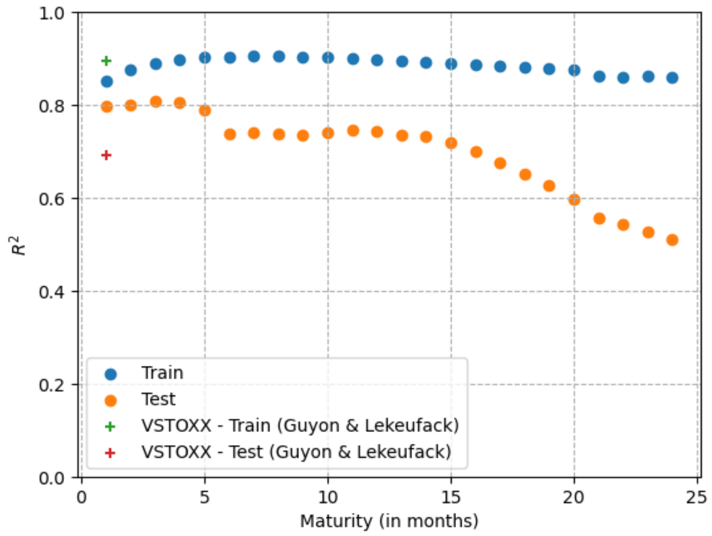
Empirical study

- We propose to check whether the **variations of the ATM IV** can be explained by the past returns and the past squared returns of the underlying asset price using Guyon and Lekeufack's model.
- **Data sets:** daily implied volatility surfaces corresponding to options on the S&P 500 index and the Euro Stoxx 50 index respectively and covering the period 08/03/2012-30/12/2022.



S&P 500



Euro Stoxx 50

Key findings:

- A large part of the movements of the ATM implied volatility can be explained by the past movements of the underlying asset price → **Implied volatility (also) is path-dependent !**
- **Long-term options are less sensitive** to the variations of the underlying asset price than short-term options.
- The gap between the scores on the train and the test sets seems to be due to the peculiar nature of the test set and the small size of the data set.

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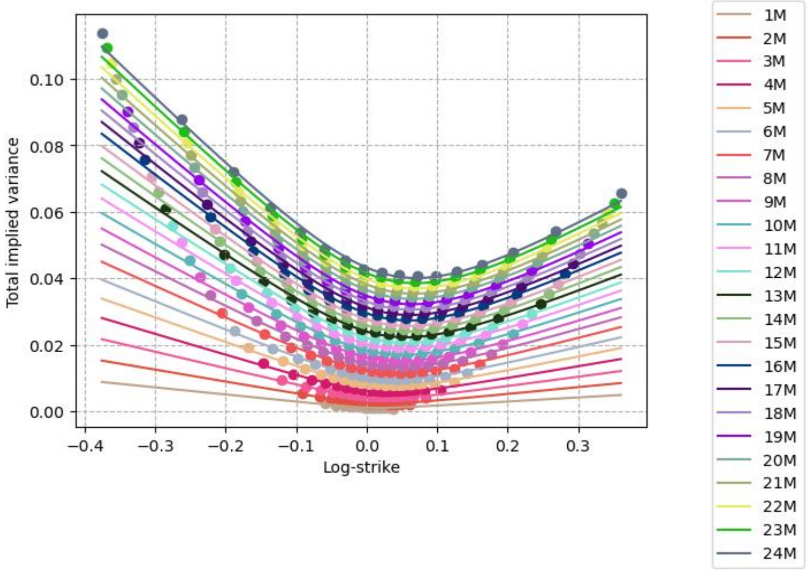
SSVI description

- Surface Stochastic Volatility Inspired (SSVI) = a popular **parametrization of the total implied variance surface** $\boldsymbol{w(k,T) = \sigma_{BS}^2(k,T) \times T}$.
  - Introduced by Gatheral & Jacquier[1] in 2014.
  - It fits market implied volatilities very well with a small number of parameters.
  - Sufficient conditions on the parameters have been found for the implied volatility surface to be **free of static arbitrage** (calendar spread and butterfly arbitrages).

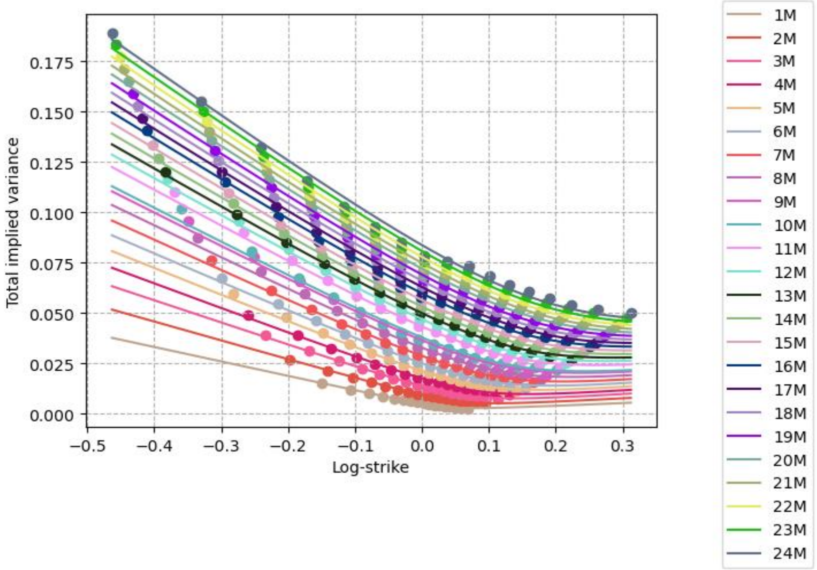
SSVI definition	SSVI parameters
$$w(k,T) = \frac{\theta_T}{2}\left(1 + \rho\varphi(\theta_T)k + \sqrt{(\varphi(\theta_T)k + \rho)^2 + (1-\rho)^2}\right)$$ where θ_T is the ATM total implied variance, φ is a smooth function from $\mathbb{R}_+^*$ to $\mathbb{R}_+^*$ such that the limit $\lim_{t \to 0} \theta_t \varphi(\theta_t)$ exists in $\mathbb{R}$.	▪ One parameter $\boldsymbol{\theta_T}$ per maturity T ▪ One parameter $\boldsymbol{\rho} \in (-1,1)$ which does not depend on the maturity ▪ A function $\boldsymbol{\varphi}$ to be specified. Several parametric forms can be considered. We compared several options and retained the following one: $$\varphi(\theta) = \frac{\eta}{\theta^\gamma (1+\theta)^{1-\gamma}}.$$ with $\gamma \in (0,1)$ and $\eta > 0$.

SSVI fit to market data

- Calibration of the SSVI by **minimization of the sum of squared errors** for each implied volatility surface in our data sets.



S&P 500 (October 19, 2016): the average relative error is equal to 1.19%



Euro Stoxx 50 (February 18, 2022): the average relative error is equal to 1.18%

In average, we obtain a **very satisfying replication** of the market IVSs: the average relative error between market and model IVs is around 1.2% over all dates.

Parsimonious SSVI

- To limit the number of parameters, we make the two following simplifications
  1. We set $\boldsymbol{\gamma} = \frac{\mathbf{1}}{\mathbf{2}}$.
  2. We assume that $\boldsymbol{\theta_T} = \boldsymbol{aT^p}$ where $a, p \geq 0$.

Definition (Parsimonious SSVI)

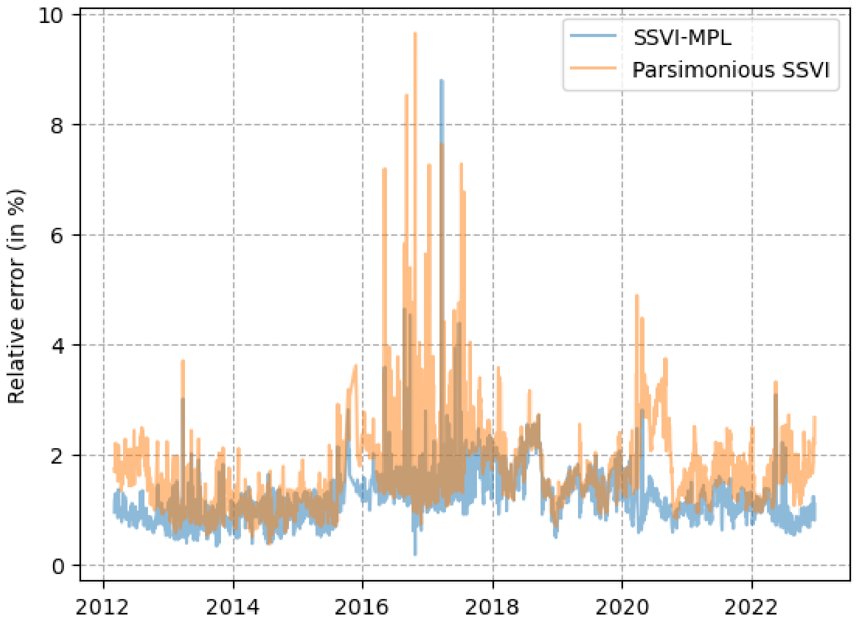
The **parsimonious SSVI** is the parameterization of the total implied volatility surface defined by :

$$w(k,T) = \frac{\theta_T}{2}\left(1 + \rho\varphi(\theta_T)k + \sqrt{(\varphi(\theta_T)k + \rho)^2 + 1 - \rho^2}\right)$$

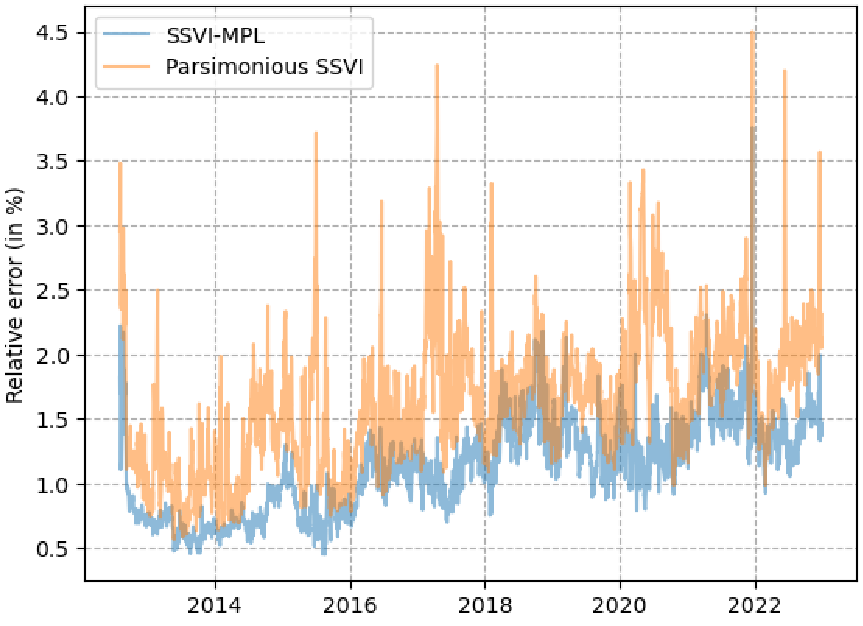
where $\theta_T = aT^p$ and $\varphi(\theta) = \frac{\eta}{\sqrt{\theta(1+\theta}}$ with $a, p \geq 0$, $\rho \in (-1,1)$ and $\eta > 0$.

Performance of the parsimonious SSVI

- Despite the very small number of parameters (**only 4!**), the replication of market IV surfaces remains satisfying with the parsimonious SSVI compared to the SSVI.



S&P 500



Euro Stoxx 50

Path-dependency of the parsimonious SSVI parameters

- **Idea:** study to which extent the PDV model explains the variations of the four parameters of the parsimonious SSVI.
- We fit the PDV model on the calibrated parameters:

$$X_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \Sigma_t$$

where $X \in \{a, p, \rho, \eta\}$.

	S&P 500		Euro Stoxx 50	
	Train	Test	Train	Test
a	91.8%	51.1%	85.7%	49.4%
p	39.4%	48.5%	76.3%	75.3%

R^2 scores of the PDV model on the parameters of the parsimonious SSVI on the train and the test sets.

Key findings:

- The evolution of the parameters a and p is **well explained** by the evolution of the underlying asset price.
- On the contrary, we have found that the evolution of the parameters ρ and η is **mostly exogenous**.

Dynamics of the underlying asset price

- The underlying asset price is assumed to evolve as follows:

$$\begin{cases} \dfrac{dS_t}{S_t} = \sigma_t dW_t^S \\ \sigma_t = \left|\beta_0^\sigma + \beta_1^\sigma R_{1,t}^\sigma + \beta_2^\sigma \Sigma_t^\sigma + \epsilon_t^\sigma\right| \\ R_{1,t}^\sigma = \displaystyle\int_{-\infty}^{t} \frac{Z_{\alpha_1^\sigma,\delta_1^\sigma}}{(t-u+\delta_1^\sigma)^{\alpha_1^\sigma}} \times \frac{dS_u}{S_u} \\ \Sigma_t^\sigma = \sqrt{\displaystyle\int_{-\infty}^{t} \frac{Z_{\alpha_2^\sigma,\delta_2^\sigma}}{(t-u+\delta_2^\sigma)^{\alpha_2^\sigma}} \times \left(\frac{dS_t}{S_t}\right)^2} \end{cases}$$

Trend feature: weighted average of past returns

Volatility feature: weighted average of past squared returns

Additive residuals allowing to account for the fact that the PDV model does not perfectly explain the variations of the spot volatility σ_t. We consider a non-central t distribution.

Dynamics of the parameters of the parsimonious SSVI

- The parameters a and p are assumed to evolve as follows:

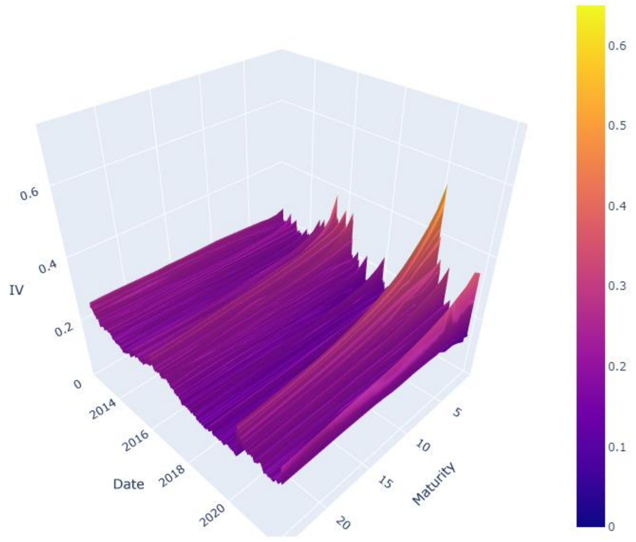
Multiplicative residuals allowing to account for the fact that the PDV model does not perfectly explain the variations of a and p

$$
\begin{cases}
a_t = \kappa_t^a(\beta_0^a + \beta_1^a R_{1,t}^a + \beta_2^a \Sigma_t^a) \\
p_t = \kappa_t^p \exp(\beta_0^p + \beta_1^p R_{1,t}^p + \beta_2^p \Sigma_t^p) \\
R_{1,t}^i = \int_{-\infty}^t \frac{Z_{\alpha_1^i,\delta_1^i}}{(t-u+\delta_1^i)^{\alpha_1^i}} \times \frac{dS_u}{S_u}, i \in \{a,p\} \\
\Sigma_t^i = \sqrt{\int_{-\infty}^t \frac{Z_{\alpha_2^i,\delta_2^i}}{(t-u+\delta_2^i)^{\alpha_2^i}} \times \left(\frac{dS_t}{S_t}\right)^2}, i \in \{a,p\}
\end{cases}
$$

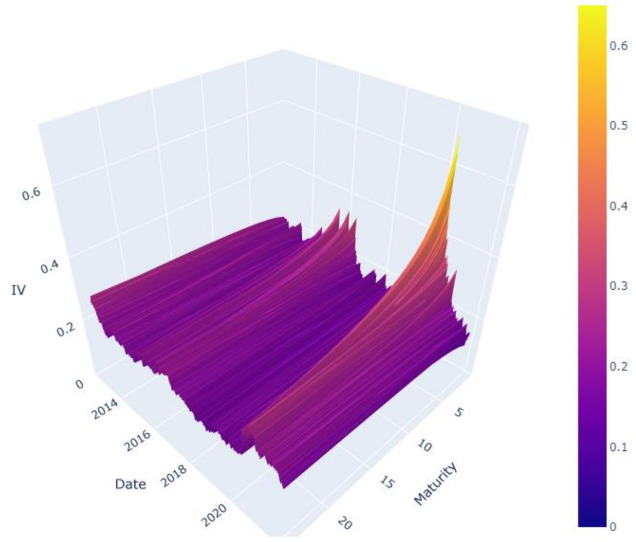
- The vector $X_t = (\kappa_t^a, \kappa_t^p, \rho_t, \eta_t)$ is modelled using a **semi-Markov diffusion model** with two states. Conditionally on each state of the semi-Markov process, $(\kappa_t^a)_{t\geq 0}, (\kappa_t^p)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ are **Cox-Ingersoll-Ross processes** while $(\rho_t)_{t\geq 0}$ is a **Jacobi process** lying between -1 and 1.

IVS sample paths

- We first simulate the parameters of the parsimonious SSVI **conditionally on the historical path** of the underlying index, i.e. S is not simulated but fixed to the historical path.



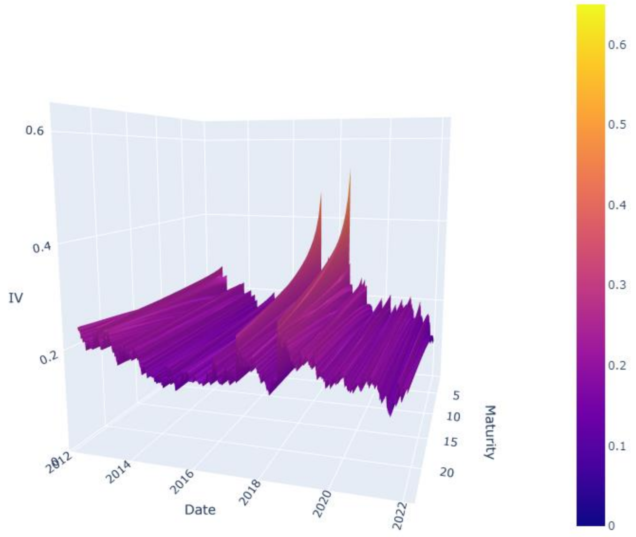
Historical path of the Euro Stoxx 50 ATM term structure



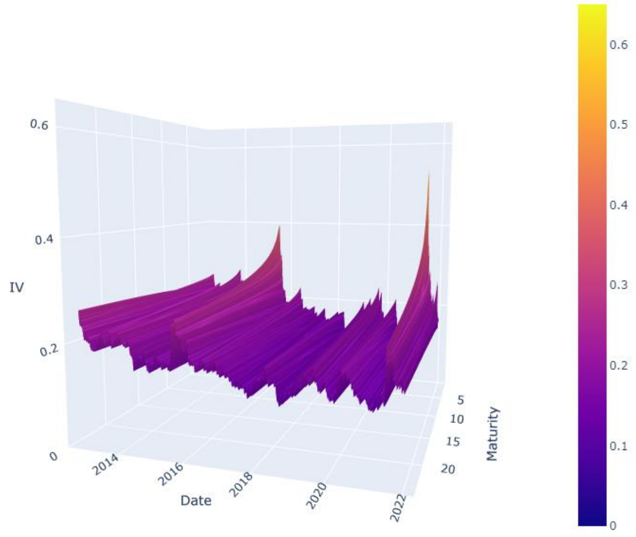
Sample path of the Euro Stoxx 50 ATM term structure

IVS sample paths

- Now, we also simulate the underlying asset price S.

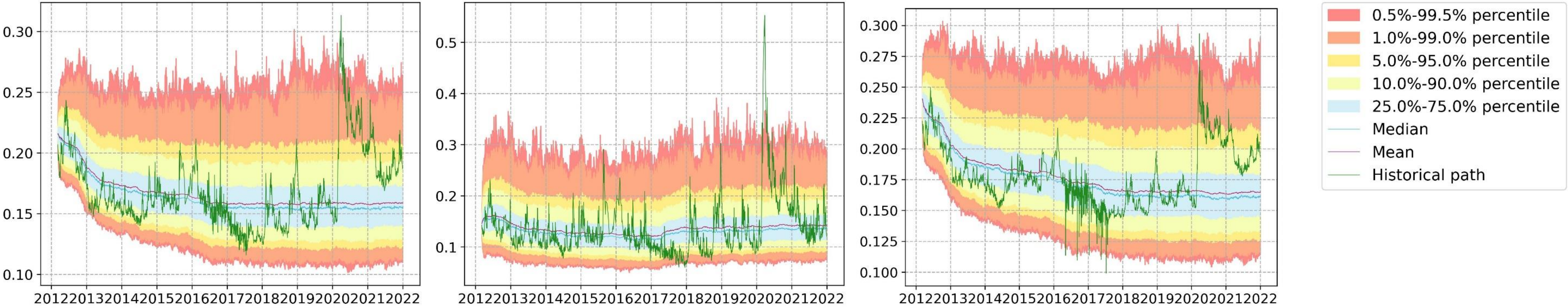


Sample path of the S&P 500 ATM term structure



Sample path of the Euro Stoxx 50 ATM term structure

Quantile envelopes



1-month ATM IV of the S&P 500 12-months ATM IV of the S&P 500 24-months ATM IV of the S&P 500

→ The simulated IVSs are highly realistic both from a pathwise and a statistical point of view

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Definition and problem statement

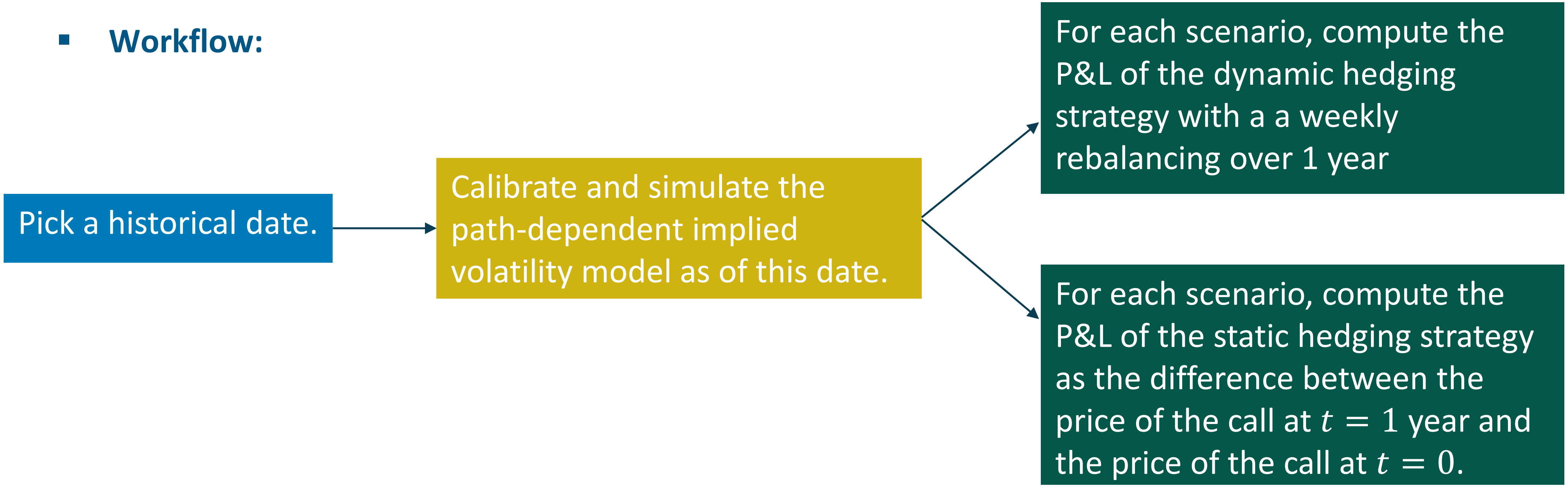
- **Registered index-linked annuities** (RILAs) = investment products offered by insurance companies that provide policyholders with returns based on the performance of an underlying index, such as the S&P 500. These products embed:
  - ➢ **An upside crediting mechanism**: participation rate, trigger rate, cap rate, etc.
  - ➢ **A downside protection option**: generally either a buffer or a floor.
- **Typical terms** of RILAs: 1-year, 3-year, 5-year and 6-year.

Payoff of a RILA with a cap rate of 25% and a buffer of -20%

- **Three hedging strategies:**
  - ➢Static hedging: the company invests in a combination of calls and puts matching the RILA's payoff.
  - ➢Dynamic hedging: the company invests in a replicating portfolio consisting of the underlying index and a risk-free bond and rebalance it on a regular basis.
  - ➢Hybrid hedging: the company switches between static vs. dynamic hedging based on the level of current market implied volatilities.

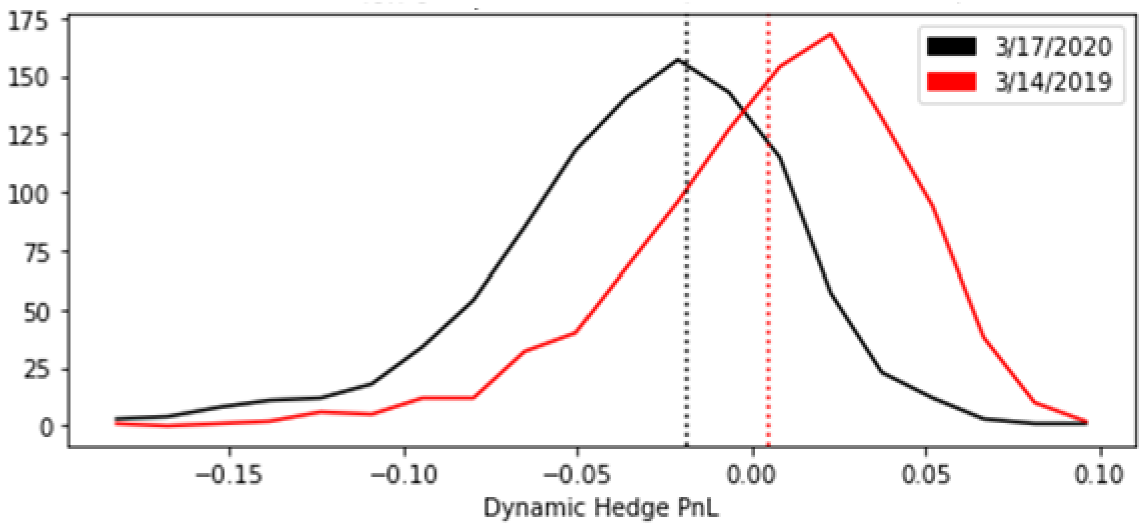
Numerical investigation of the hybrid hedging strategy

- **Setting:** 6-year term RILA product with a cap rate of 25% and a buffer of -20%
- **Static hedging:** buy a 6-year ATM call, short a 6-year 125% OTM call and a 80% OTM put.
- **Objective:** compare the P&L of the static hedging with the strategy consisting of dynamically hedging the 6-year ATM call over the 1st year and then buy a 5-year ATM call.
- **Workflow:**



Numerical investigation of the hybrid hedging strategy

- The use of stochastic scenarios allows to get a **distribution** of the difference between the two P&L's.



- Based for example on the average difference, **one can decide whether it is more profitable to use a static or dynamic hedging strategy**.
- At this stage, we have done this exercise for many dates and **the model allowed to take the right decision most of the time**.

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Key takeaways

- We introduce a new model allowing to **jointly simulate implied volatility surfaces and the underlying asset price.**

- The model produces **highly realistic paths** allowing to improve the risk assessment in applications requiring implied volatility scenarios (cf hedging application).

- More generally, path-dependent models appear as a **promising area of research** for real-world financial modelling.

Thank you

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