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On the Linearization of Bilinear Mortality Models

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Lee–Carter model

$$\ln m_{x,t} = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}.$$

- **Advantages:** simple and parsimonious representation of age–time mortality dynamics.
- **Unidentifiability:** the parametrization is not unique. Hence, constraints are required for identification. Otherwise, the parameters cannot be interpreted meaningfully and statements such as “ β_x measures age sensitivity” are not well-defined.
- **Estimation limitations:** The original LC model cannot be fit as a regular regression model due to the lack of covariates, and, in the original paper, a Singular Value Decomposition (SVD) was applied to fit the model (Lee and Carter [5]).
- **Bilinearity:** the model is fundamentally bilinear, with β_x and κ_t entering multiplicatively.

This Talk

We propose a general framework that transforms the bilinear structure of the Lee–Carter model and a broad class of its extensions into a fully linear form, enabling the direct use of standard linear modeling tools.

- Retains the simplicity and parsimony of Lee–Carter, while improving forecast accuracy.
- Identifiable under standard full-rank conditions of the design matrices.
- Can be estimated as a regular regression model.
- Parameters estimated via generalized estimating equations (GEE), allowing principled modeling of serial dependence (Liang and Zeger [7]).

Estimation of the Lee–Carter Model via SVD

Let Y denote the matrix of log-mortality rates, with rows corresponding to years and columns to ages.

Estimation of α

$$\hat{\alpha}_x = \frac{1}{T} \sum_{t=1}^T \ln m_{x,t}.$$

Define the centered mortality surface

$$\tilde{Y}_{t,x} = \ln m_{x,t} - \hat{\alpha}_x.$$

Apply the singular value decomposition

$$\tilde{Y} = UDV^T.$$

Estimation of β and κ

$$\hat{\beta}_x = \frac{v_{x1}}{\sum_{i=1}^A v_{i1}}, \quad \hat{\kappa}_t = d_1 u_{t1} \sum_{i=1}^A v_{i1}.$$

The time index as the first principal component

- Principal component analysis (Abdi and Williams [1]) of the centered mortality matrix \tilde{Y} is based on the sample covariance matrix

$$\frac{1}{T} \tilde{Y}^T \tilde{Y} = V \left(\frac{D^2}{T} \right) V^T.$$

- The principal component score vectors satisfy

$$\tilde{Y} v_{.j} = d_j u_{.j}.$$

- Consequently, the first principal component score coincides with the Lee–Carter period index (up to sign and constraints).

PCA–GEE single-population model

Step 1: Extraction of the time index

$$k_t \equiv \text{PC1 score}_t = d_1 u_{t1}.$$

Step 2: GEE conditional on k_t

Conditioning on the externally extracted mortality index k_t , the single-population PCA–GEE model is

$$\ln m_{xt} = \alpha_x + \beta_x k_t + \gamma(t - x) + \varepsilon_{xt}, \quad \varepsilon_x \sim \mathcal{N}(\mathbf{0}, \sigma^2 R(\alpha)).$$

- α_x : age-specific intercept
- β_x : age-specific slope
- $\gamma(t - x)$: cohort effects
- $R(\alpha)$: working correlation (serial dependence over time)

Estimation via GEE

The PCA-GEE model is estimated using generalized estimating equations via `geeglm` (Højsgaard, Halekoh, and Yan [4]):

$$\text{geeglm}\left(\ln m_{xt} \sim -1 + \text{age} + \text{age}:k_t + \text{cohort}, \text{id} = \text{age}, \text{waves} = \text{year}, \text{corstr} = \text{"ar1"}\right).$$

- `age`: factor
- `cohort`: $(t - x)$
- `id`, `waves`: clustering by age with time ordering
- `corstr`: AR(1) working correlation for serial dependence

Empirical analysis

Data

- 56 populations from the Human Mortality Database
- Annual age-specific mortality (1×1), ages 0–90
- Training period: 1961–2000
- Forecast evaluation: 2001 onward

Competing models

- **Model 1:** Original bilinear Lee–Carter
- **Model 2:** Linearized Lee–Carter (via PCA–GEE)
- **Model 3:** Linearized Lee–Carter with cohort effect

Model 1: Bilinear Lee-Carter

$$\ln m_{xt} = \alpha_x + \beta_x \kappa_t + \varepsilon_{xt}, \quad \varepsilon_{xt} \sim \mathcal{N}(\mathbf{0}, \sigma^2).$$

Model 2: Linearized Lee-Carter

$$\ln m_{xt} = \alpha_x + \beta_x k_t + \varepsilon_{xt}, \quad \varepsilon_x \sim \mathcal{N}(\mathbf{0}, \sigma^2 R(\alpha)).$$

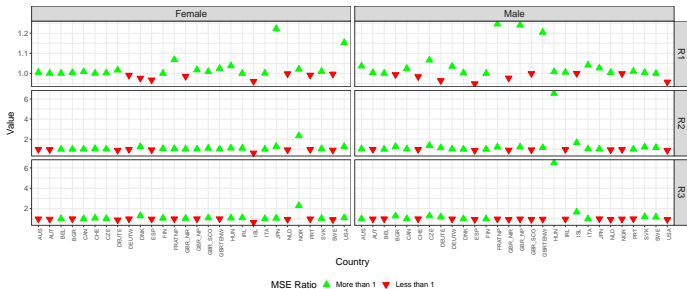
Model 3: Linearized Lee-Carter with cohort effect

$$\ln m_{xt} = \alpha_x + \beta_x k_t + \gamma(t - x) + \varepsilon_{xt}, \quad \varepsilon_x \sim \mathcal{N}(\mathbf{0}, \sigma^2 R(\alpha)).$$

Forecasting performance

- Mortality indices are projected using random walks with drift.
- Coefficients are assumed constant over the forecast horizon.
- Accuracy assessed using out-of-sample MSE (2001–latest)

$$R_1 = \frac{\text{MSE}_{\text{LC}}}{\text{MSE}_{\text{PCA-GEE}}}, R_2 = \frac{\text{MSE}_{\text{LC}}}{\text{MSE}_{\text{PCA-GEE}+\text{C}}}, R_3 = \frac{\text{MSE}_{\text{PCA-GEE}}}{\text{MSE}_{\text{PCA-GEE}+\text{C}}}.$$



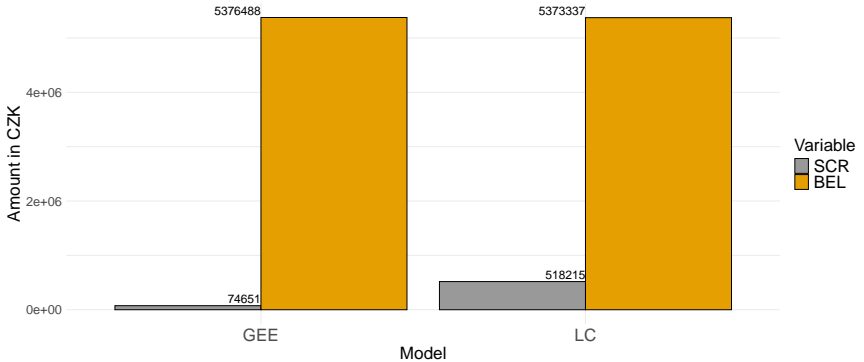
Actuarial applications

- **Best Estimate Liability (BEL):** the unbiased estimate of the present value of expected future cash flows, calculated under best-estimate assumptions without explicit margins.
- **Solvency Capital Requirement (SCR):** the amount of capital the insurance company needs to hold in order to survive a 1 in 200 event (Dickson, Hardy, and Waters [3] and Börger [2]).

Empirical setup

- Hypothetical portfolio of 20 individuals with specified year of birth and premiums denominated in CZK
- Czech mortality data, ages 41–90, years 1950–2019
- Long-term mortality projections over a 56-year horizon
- Valuation year: 2025 Interest rate: 1%
- Retirement age: 67 Maximum age: 90
- Annual pension benefit: CZK 25,000

Comparison of BEL and SCR



Stochastic dynamics

- Lee–Carter: estimated variance ≈ 1.84
- PCA–GEE: estimated variance ≈ 0.04

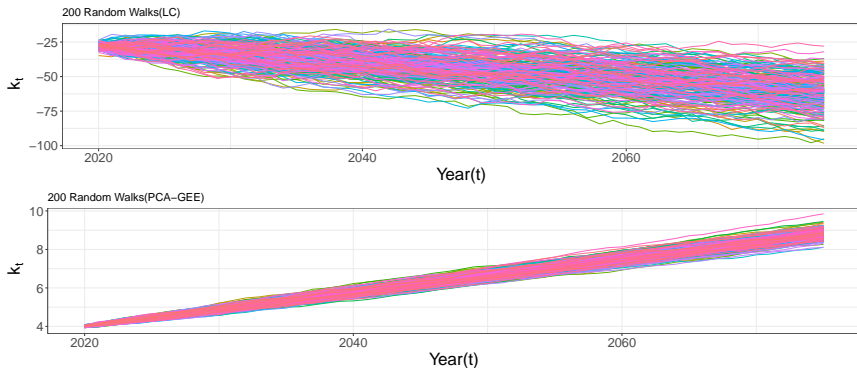


Figure: Simulated trajectories of the mortality index k_t under Lee–Carter and PCA–GEE specifications.

PCA–GEE Multi-Population Mortality Model

$$\ln m_{cgxt} = \alpha_{cgx} + \beta_x k_t + \beta_{cgx} k_{cgt} + \gamma(t - x) + \varepsilon_{cgxt},$$

$$\varepsilon_{cgx} \sim \mathcal{N}(\mathbf{0}, \sigma^2 R(\alpha)).$$

- k_t : common mortality factor
- k_{cgt} : population-specific factor
- $\gamma(t - x)$: cohort effects (optional)
- $R(\alpha)$: working correlation (AR(1) or exchangeable)

Estimation: Generalized Estimating Equations (GEE)

Empirical Study: Austria and Czech Republic (1991–2021)

Training: 1991–2010

Forecast: 2011–2021

Ages: 0–90 Genders: Female, Male

Model 1: Bilinear Li–Lee(Li and Lee [6])

$$\ln m_{cgt} = \alpha_{cgx} + \beta_x \kappa_t + \beta_{cgx} \kappa_{cgt}$$

(Joint SVD estimation)

Model 2: PCA–GEE (Linearized)

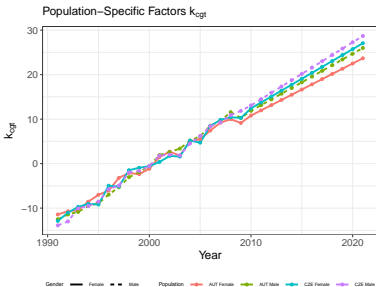
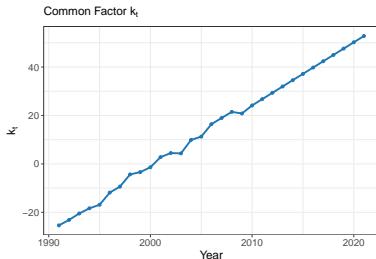
$$\ln m_{cgt} = \alpha_{cgx} + \beta_x k_t + \beta_{cgx} k_{cgt}$$

Model 3: PCA–GEE + Cohort

$$\ln m_{cgt} = \alpha_{cgx} + \beta_x k_t + \beta_{cgx} k_{cgt} + \gamma(t - x)$$

Forecasting the Mortality Factors

- Models estimated over 1991–2010
- k_t and k_{cgt} modeled as random walks
- Multi-step forecasts generated for 2011–2021
- Forecasted factors inserted into linear predictors



Forecast Accuracy Comparison

- Evaluation metric: Mean Squared Error (MSE)
- Benchmark: Bilinear Li–Lee model
- PCA–GEE models estimated with AR(1) or Exchangeable correlation

Key Findings:

- PCA–GEE models outperform bilinear Li–Lee in most cases
- Modeling serial dependence improves stability
- Separation of common and population-specific factors enhances forecasting accuracy

Conclusion: Linearization + GEE \Rightarrow More accurate and stable mortality forecasts.

Future Research Directions

- Extension to additional latent factors within the linear framework.
- Time-varying coefficient specifications, relaxing the assumption that α_x and β_x are constant over time.
- Joint modeling of mortality with related economic processes (e.g., incorporating GDP per capita to capture the link between development and mortality).

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Thank You for Your Attention!