Efficient computation of Solvency Capital Requirement using Multilevel Monte-Carlo methods

Mathieu Truc, Milliman

About the speaker

- **Mathieu Truc –** *R&D Consultant, PhD Candidate, Milliman*

  Mathieu Truc joined Milliman R&D team in Paris in January 2023 after a 6 months internship. In partnership with Sorbonne University and Milliman he began a PhD thesis in May 2023 around the topic of numerical methods for economic capital estimation.

- Milliman is a global actuarial firm with expertise in Health, Insurance, Retirement & Benefits and Risk

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Outline of the talk

1. Solvency Capital Requirement (SCR) estimation
   a. Challenges
   b. Nested Monte-Carlo

2. Introduction to Multi-Level Monte-Carlo methods

3. Numerical experiments
   a. A very simple model
   b. A more realistic Asset-Liabilities Management model

The SCR estimation problem

- A simplified Solvency II (SII) balance-sheet :

Assets	Liabilities
MV_t	NAV_t
	BEL_t

Market-Value of the company assets

Net Asset Value: $NAV_t = MV_t - BEL_t$

Best-Estimate of Liabilities

- Market Consistent Valuation of the Net Asset Value :

$$NAV_t = \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\sum_{u=t}^{T} D(t,u)C_u}_{NPV_{t:T}} \middle| X_t\right]$$

Discount factor between year t and u

Market information at year t

Margin cash-flow at year u

The SCR estimation problem

- Typical Monte-Carlo valuation of NAV_t :

Risk-Neutral scenarios generation → Net Present Value (NPV) valuation → Empirical averaging

Index	Mat	t	$t+1$	…	T
Stock		1	1.05	…	413
ZC Bond	1	.96	.98	…	.95
…	….	….	….	….	….
ZC Bond	30	.002	.0025	…	.005

$\times K$

Cash-Flows	$t+1$	…	T	Discounted Σ
Assets	150	…	75	1000
Liabilities	90	…	85	995
Margin	60	…	-10	**5**

$\times K$

$$NAV_t \approx \frac{1}{K}\sum_{k=1}^{K} NPV_{t:T}^{i}$$

The SCR estimation problem

- Definition of the Solvency Capital Requirement (SCR) :

$$SCR_t = NAV_t + x_t^\star$$

➔ $x_t^\star$ is the minimum amount of extra capital to set aside today to insure solvency with probability of 99.5%

- 1-year loss in Net Asset Value :

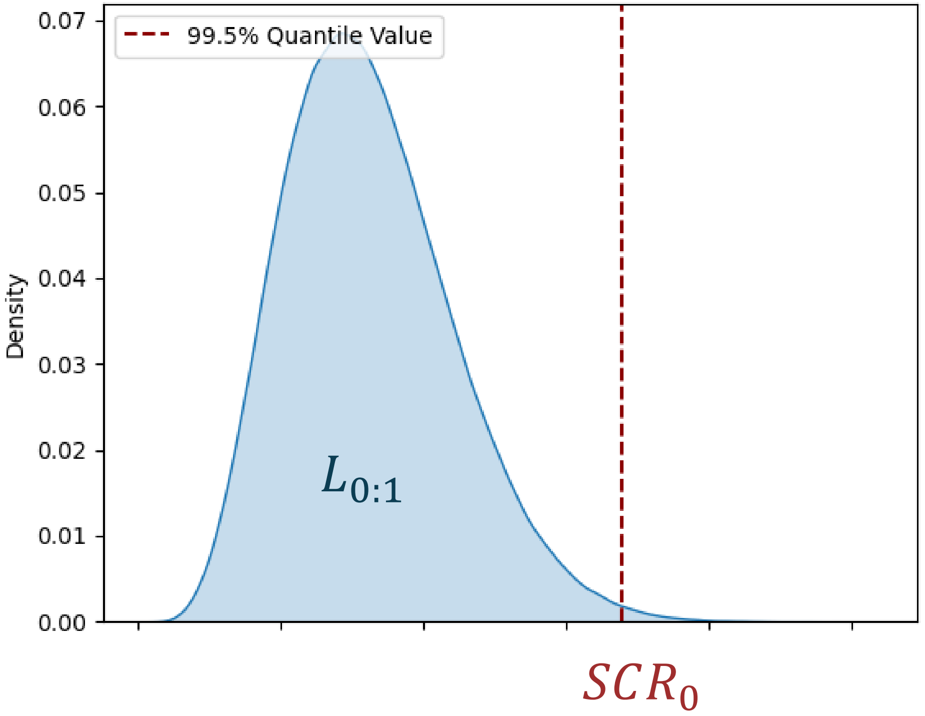
$$L_{t:t+1} = NAV_t - NAV_{t+1}$$

Random variable at year t

- Expression as a quantile :

$$SCR_t = q_{99.5\%}^t(L_{t:t+1})$$

Quantile on the 1-year loss



The SCR estimation problem

- Problem :

$$L_{t:t+1} = NAV_t - NAV_{t+1} = NAV_t - \mathbb{E}^{\mathbb{Q}}\left[NPV_{t+1:T} \middle| X_{t+1}\right]$$

➔ $L_{t:t+1}$ cannot be sampled exactly

- Proxy methodology :
  - Theory gives that : $$\Psi(X_{t+1}) = \mathbb{E}^{\mathbb{Q}}\left[NPV_{t+1:T} \middle| X_{t+1}\right]$$
  - Construct a proxy :

$$\widehat{\Psi}(X_{t+1}) \approx \Psi(X_{t+1})$$

- Non-Proxy methodology :
  - NAV_{t+1} can be sampled approximately by a Monte-Carlo procedure

Proxy Methods vs Non-Proxy Methods

- Proxy Methods :
  - Currently the most popular methodologies
  - Can be hard to calibrate under stressed market conditions or complex response function
  - Cumbersome validation of the proxy

- Non-Proxy Methods:
  - May require high-computational capabilities but strong parallel computing possibilities
  - No deterioration for complex insurance portfolios
  - No proxy to validate

(Crude) Nested Monte-Carlo

- Approximate sampling of :

$$L_{0:1} = NAV_0 \ - \mathbb{E}^{\mathbb{Q}}\left[NPV_{1:T} \middle| X_1\right]$$

- Idea : $\mathbb{E}^{\mathbb{Q}}\left[NPV_{1:T} \middle| X_1\right]$ can be approximated by Monte-Carlo

$$\hat{E}_K(X_1) = NAV_0 \ - \frac{1}{K}\sum_{k=1}^{K} NPV_{1:T}^{k}(X_1)$$

i.i.d sample of $NPV_{1:T}$ conditionally to X_1

- Approximate sample of $L_{0:1}$:
  1. Sample :

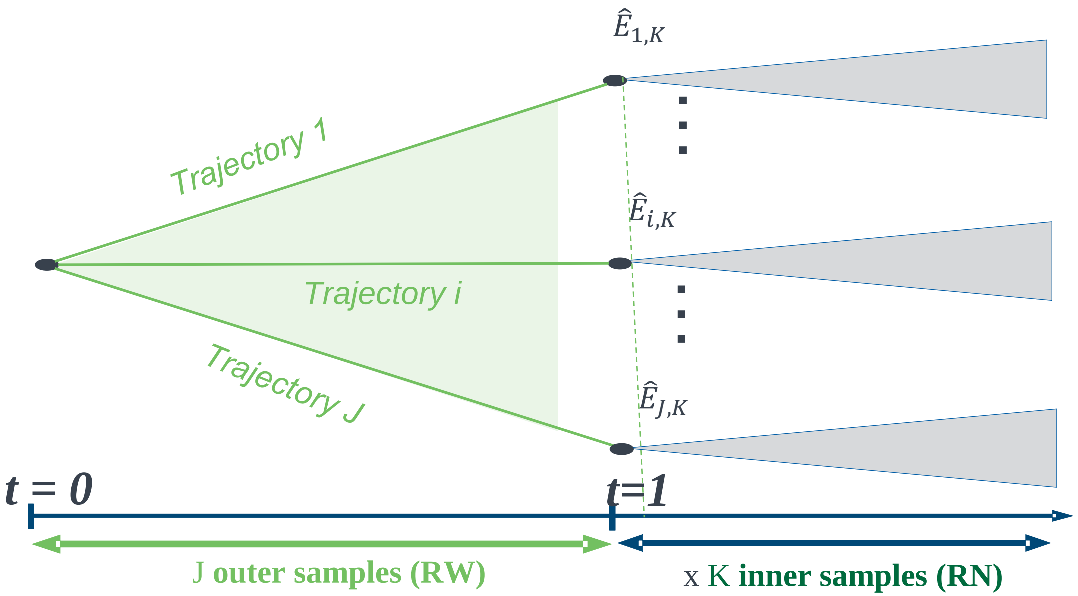
$$(X_1^1, \dots, X_1^J)$$

i.i.d sample of X_1

  2. Compute :

$$\left(\hat{E}_K(X_1^1), \dots, \hat{E}_K(X_1^J)\right)$$

Approximate sample of $L_{0:1}$



(Crude) Nested Monte-Carlo

- Intuition of the method :

$$\mathbb{P}(L_{0:1} \geq \eta) \approx \mathbb{P}\left(\hat{E}_K(X_1) \geq \eta\right) = \mathbb{E}\left[\mathbb{I}_{\hat{E}_K(X_1)\geq\eta}\right] \approx \frac{1}{J}\sum_{j=1}^{J} \mathbb{I}_{\hat{E}_K\left(X_1^j\right)\geq\eta}$$

- Interpretation of the parameters :
  - J the number of outer samples control the Variance of the estimation
  - K the number of inner samples control the Bias of the estimation

- Optimal balance, see ([1] and [2]):
  - For a computational budget Γ

$$J = C_1\Gamma^{2/3} \qquad K = C_2\Gamma^{1/3}$$

  - Divide by 2 the estimation error ➔ 8 times more simulations
  - “A priori” numerical investigations can be performed to estimate C_1 and C_2

Multilevel Monte-Carlo Methods (MLMC)

- Introduced in [3] for the discretization of sample path with a broad literature on the subject available :
  - e.g [4] in the context of risk management
  - e.g [2] for weighted multi-level variants
- Idea : Use multiple levels of sample approximation
  - Increasing sequence of inner samples : $K_1 < K_2 = 2K_1 < \ldots < K_L = 2^L K_1$

Level 1 | Level 2 | ⋯ | Level L

$$L_{0:1} \approx \quad \hat{E}_{K_1}(X_1) \;\vdots\; \hat{E}_{K_2}(X_1) \;\vdots\; \ldots \;\vdots\; \hat{E}_{K_L}(X_1)$$

Increasing quality of approximation but increasing sampling cost

Multi-Level Monte Carlo methods (MLMC)

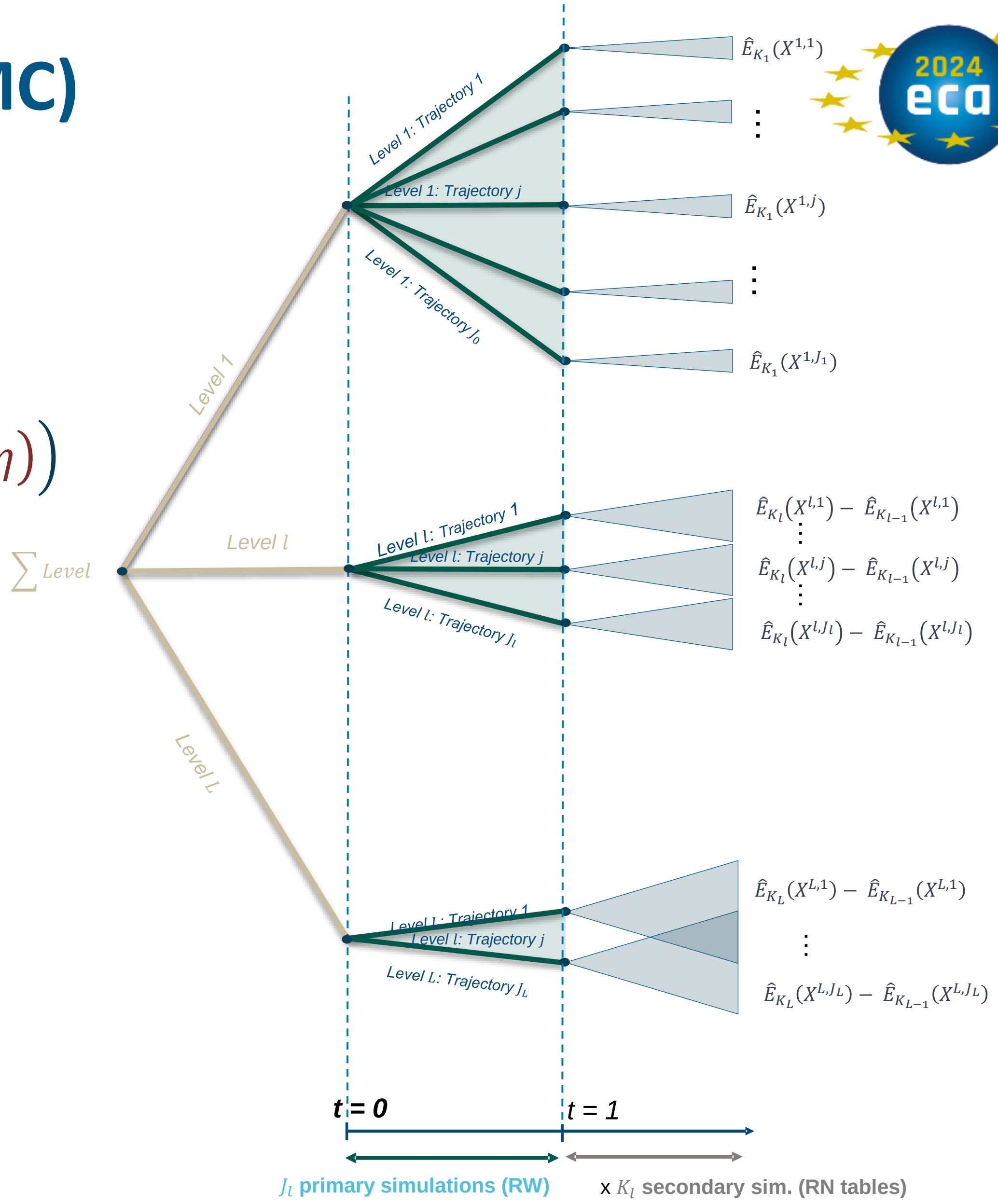
- Leverage the multiple levels with a telescopic sum :

$$\mathbb{P}(L_{0:1} \geq \eta) \approx \mathbb{P}\left(\hat{E}_{K_L}(X_1) \geq \eta\right)$$

$$= \mathbb{P}\left(\hat{E}_{K_1}(X_1) \geq \eta\right) + \sum_{l=2}^{L} W_l \left(\mathbb{P}\left(\hat{E}_{K_l}(X_1) \geq \eta\right) - \mathbb{P}\left(\hat{E}_{K_{l-1}}(X_1) \geq \eta\right)\right)$$

One Nested Monte-Carlo per level

- Optimal performances (see [2]):
  - Non weighted case (MLMC) : Dividing by 2 the estimation error ➔ 5.7 times more simulations
  - Weighted case (ML2R) : Dividing by 2 the estimation error ➔ 4 to 5 times more simulations



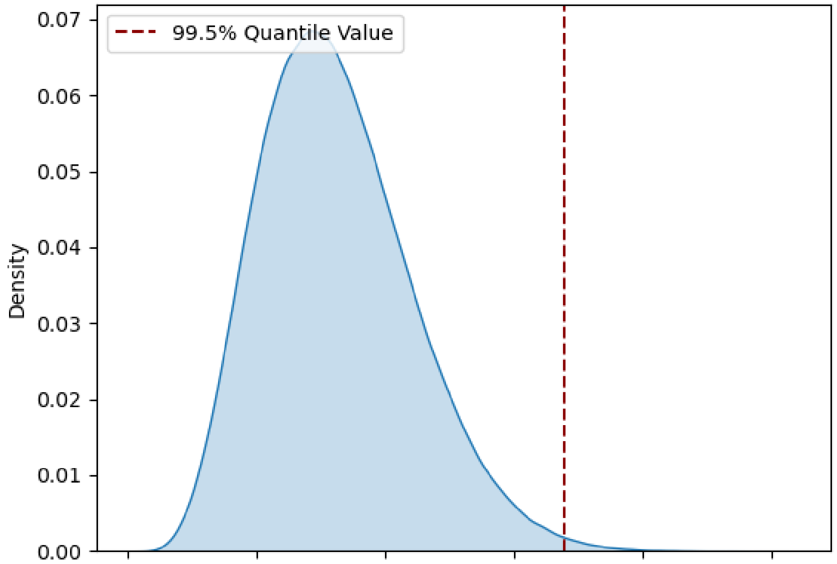
A very simple model (see [5])

- Assets of the company : Stock Index
- Liabilities of the company : Minimum guaranteed rate r_G on an initial deposit paid at horizon T

$$L_{0:1} = \mathbb{E}\left[e^{-r(T-1)}(G - S_T)^+ \middle| S_1\right]$$

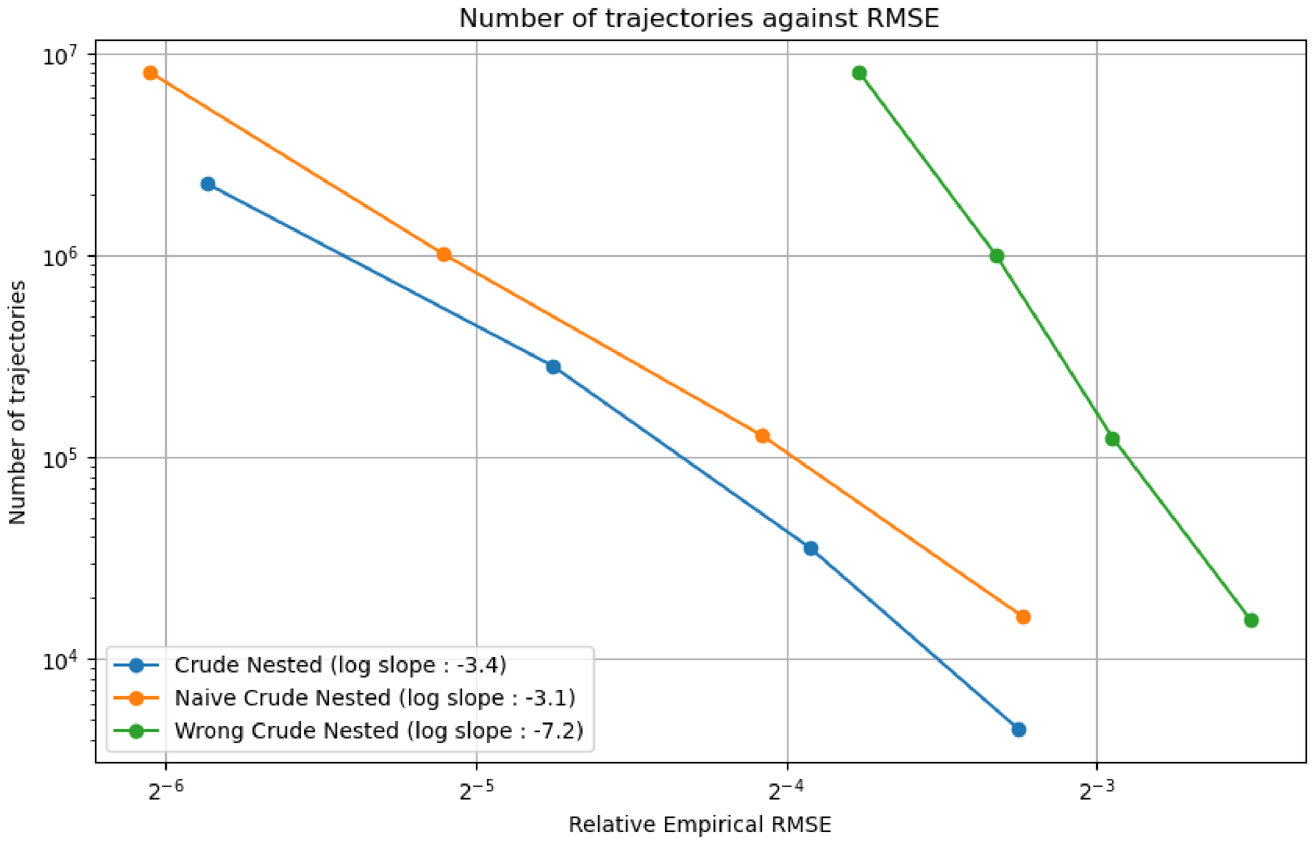
Where $G = G_0 e^{r_g T}$ is the initial deposit G_0 appreciated by the minimum guaranteed rate and

- Asset and risk-free rate model : Black-Scholes Setup
- Aim at computing $q_{99.5\%}(L_{0:1})$



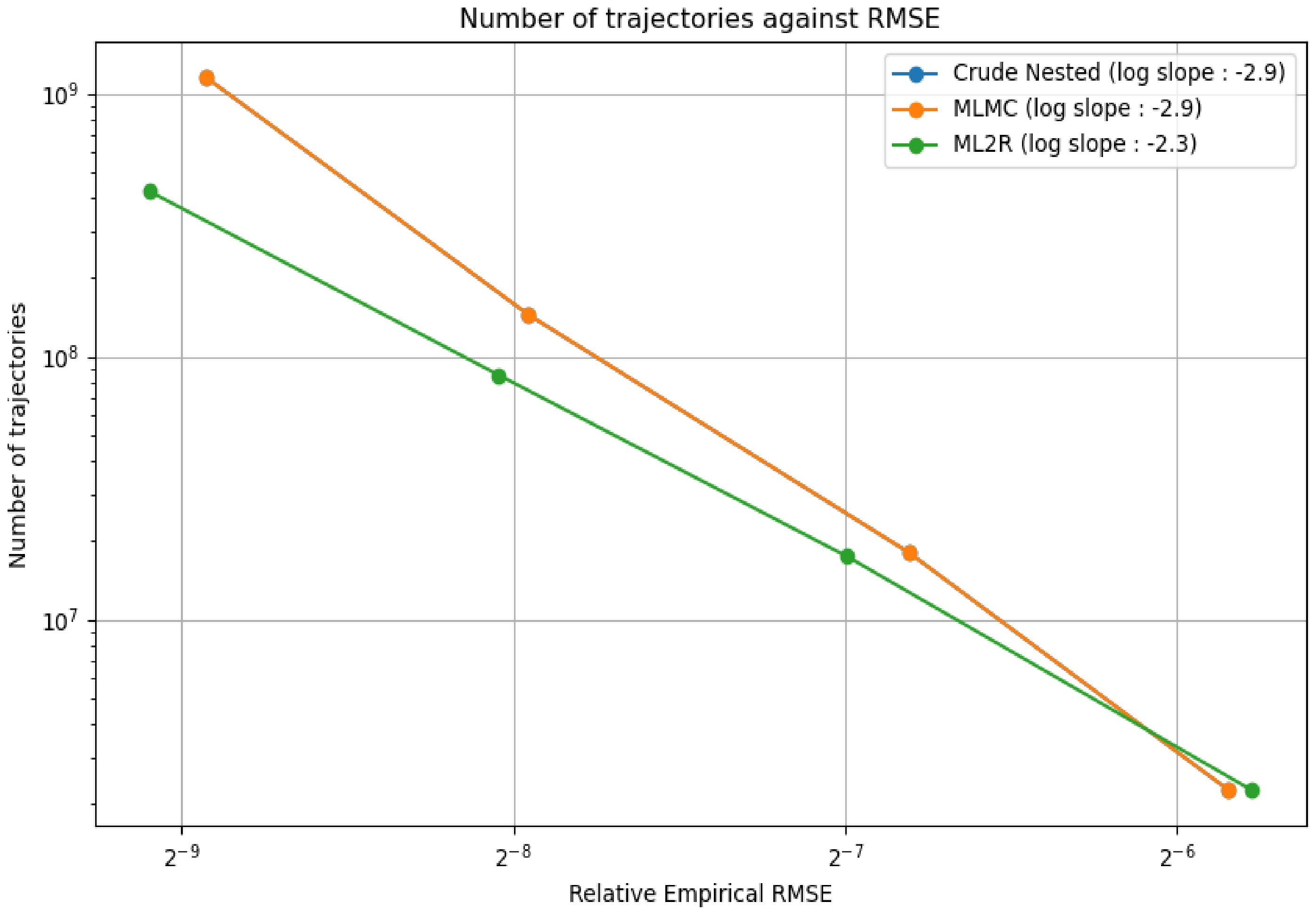
A very simple model (see [5])

- Wrong way of doing Nested Monte-Carlo
  - Using more inner samples K than outer samples J leads to catastrophic performances (wrong allocation of computational budget)
  - Taking naïve constants in the budget repartition leads to reduced performances compared when constants are estimated accurately



A very simple model (see [5])

- Multi-level performances
  - From a relative precision of 1% and higher the Weighted Multilevel method uses less trajectories
  - Dividing by almost 3 the number of trajectories for the highest precisions
  - The weights are crucial for the method to perform



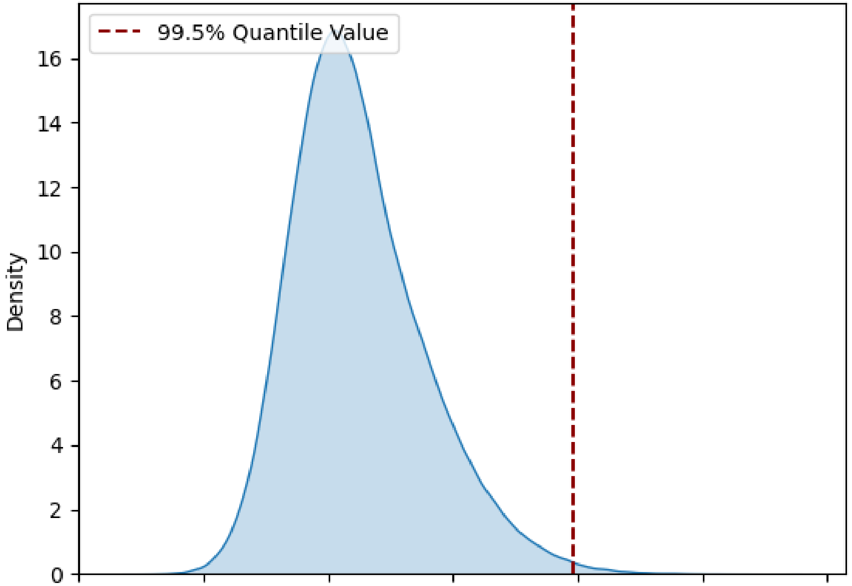
A more realistic Asset-Liabilities model (see [6])

- 1-year Loss in Net Asset Value :

$$L_{0:1} = -\mathbb{E}^{\mathbb{Q}}\left[NPV_{1:T} \middle| \mathcal{F}_1\right]$$

- We aim at estimating $\mathrm{SCR}_0 = q_{99.5\%}(L_{0:1})$

- Main features of the ALM model :
  - Basket of bonds + Stock portfolio with targeted allocations
  - Dynamic Lapses
  - Dynamic crediting rate with minimum guaranteed rate



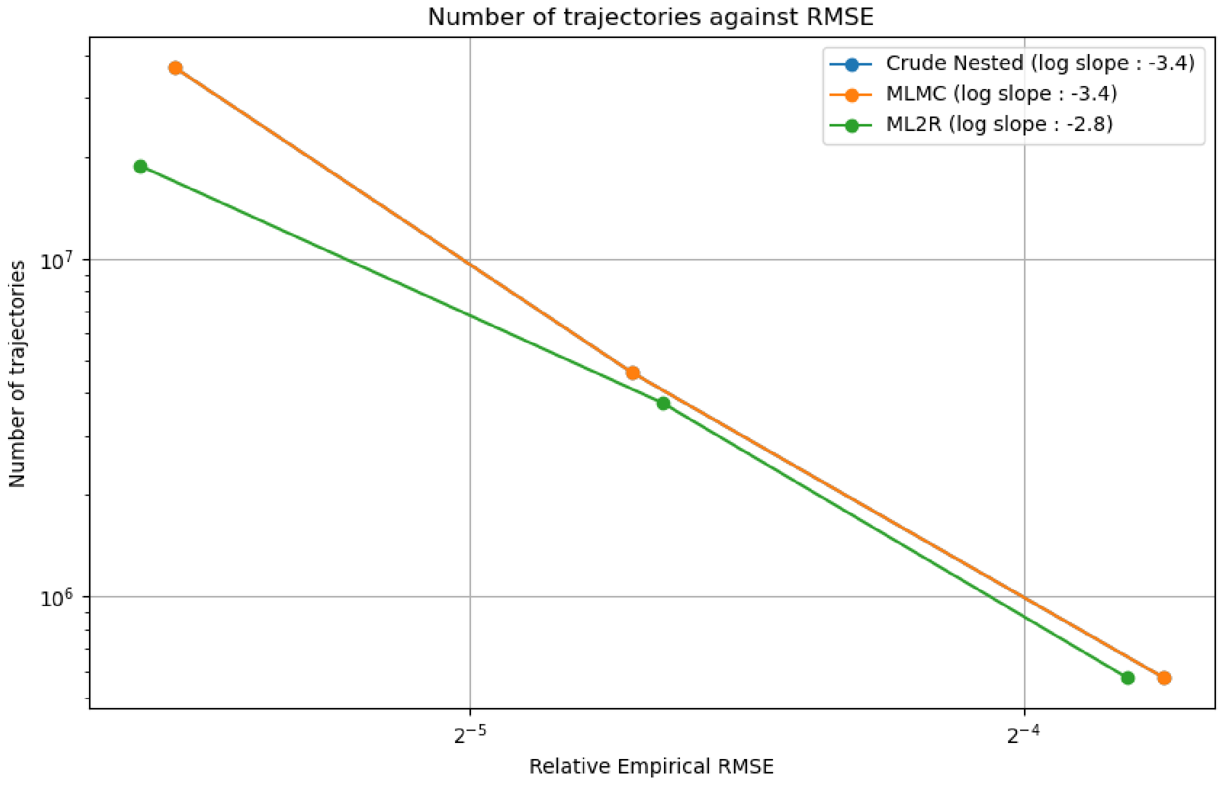
A more realistic Asset-Liabilities model (see [6])

- Multi-level performances

  - From a relative precision of 3% and higher the Weighted Multilevel method uses less trajectories

  - Dividing by almost 2 the number of trajectories for the highest precisions

  - The weights are crucial for the method to perform



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Thank you

Contact Details

- E-Mail : mathieu.truc@milliman.com
- LinkedIn : www.linkedin.com/in/mathieu-truc/