What is the value of the annuity market?

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March 2024

Jointly with Mogens Steffensen, published in Decisions in Economics and Finance

UNIVERSITY OF COPENHAGEN

Interesting Questions

Whether and when to annuitize?

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Should one put wealth at stake for mortality credits?

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What is the maximum value of the annuity market to an annuitant?

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What is the maximum value of the annuity market to an annuitant?

How does the annuity market's value depend on various market parameters of the market and mortality?

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A continuous-time life-cycle model

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Choose optimal consumption and investment.

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Calculate the individual's lifetime utility in markets without and with access to annuitization.

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A continuous-time life-cycle model

Choose optimal consumption and investment.

Calculate the individual's lifetime utility in markets without and with access to annuitization.

Compare lifetime utility measures by translating them into wealth proportions using certainty equivalents.

Focus on the Decumulation Phase

Objective: Calculate lifetime consumption in the market with and without access to an annuity

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Accumulation Phase	Decumulation Phase

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Accumulation Phase	Decumulation Phase
Low mortality rates	Higher mortality rates

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Non-hedgeablilty of labor income	

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	Decumulation Phase
	No uncertain income Explicit solution

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Other solutions: Numerical or sufficiently complete market.

The Four Problems

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Invest proportion $\pi(t)$, consumes at rate $c(t)$ at time t.

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CRRA and exponential discounting

$$u(t, s, c) = \quad e^{-\rho(s-t)} \frac{1}{1-\gamma} c^{1-\gamma}.$$

Without Insurance

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Wealth

$$\begin{aligned} &+ \quad \text{Interest} + \text{Return on the risky asset} - \text{Consumption} \\ &\pm \quad \text{The risk associated with the risky asset} \end{aligned}$$

Without Insurance

Wealth

$+$ Interest $+$ Return on the risky asset $-$ Consumption

$\pm$ The risk associated with the risky asset

$$\begin{aligned} dX(t) &= X(t)(r + \pi(t)(\alpha - r))dt - c(t)dt + X(t)\pi(t)\sigma dW(t) \\ X(0) &= x_0 > 0. \end{aligned}$$

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Uninsured individual with time-additive preferences

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$+$ **Mortality credits**

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Wealth

$$
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&\pm \quad \text{The risk associated with the risky asset} \\
&+ \quad \textbf{Mortality credits}
\end{aligned}
$$

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dX(t) &= X(t)(r + \pi(t)(\alpha - r))dt - c(t)dt + X(t)\pi(t)\sigma dW(t) \\
&+ \boldsymbol{\mu^*(t)X(t)}dt, \qquad X(0) = x_0 > 0.
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Imagine we remove away all the risk: $\mu = 0, \alpha = r, \sigma = 0$

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There is no risk left, but there is still a γ,
Aversion towards the variation of consumption over time

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Time inconsistency is dealt with by equilibrium theory.

The Four Problems

	Time-additive preferences	Separated preferences
Uninsured		
Insured		

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Uninsured	The u function (γ), ~~Mortality credits~~	
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Insured	The *u* function (γ) $+$ Mortality credits	

The Four Problems

	Time-additive preferences	Separated preferences
Uninsured	The *u* function (γ), ~~Mortality credits~~	The *u* and *v* function (γ and ϕ), ~~Mortality credits~~
Insured	The *u* function (γ) $+$ Mortality credits	The *u* and *v* function (γ and ϕ) $+$ Mortality credits

The Solutions

All solutions: $V^{\text{Prop}}(t,x) = (a^{\text{Prob}}(t))^{\text{Preferences}} \cdot \frac{1}{1-\gamma} x^{1-\gamma}$

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	Time-additive preferences	Separated preferences
Uninsured	$V^{ua}(t,x)$	$V^{us}(t,x)$
Insured	$V^{ia}(t,x)$	$V^{is}(t,x)$

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Insured	$V^{ia}(t,x)$ $a^{ia}(t)$	$V^{is}(t,x)$ $a^{is}(t)$

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The relative loss of wealth from losing access to the insurance market

The Comparison

$$V^{ia}(0, x(1-\epsilon)) = V^{ua}(0, x),$$
$$V^{is}(0, x(1-\epsilon)) = V^{us}(0, x).$$

The relative loss of wealth from losing access to the insurance market

$$\epsilon = 1 - \left(\frac{a^{ua}(0)}{a^{ia}(0)}\right)^{\frac{\gamma}{1-\gamma}},$$
$$\epsilon = 1 - \left(\frac{a^{us}(0)}{a^{is}(0)}\right)^{\frac{\phi}{1-\phi}}.$$

Numerical Results

Table: The parameters used in the numerical examples. Note r, α, and σ are thought of as corrected for inflation.

Parameters	Description	Value
z	Age at initialization/retirement	65
ρ	Impatience factor for all states	0.02
r_0	The constant drift of the risk-free asset	0.02
α	The constant drift of the risky asset	0.05
σ	The constant volatility of the risky asset	0.2
A	Parameter for pricing mortality intensity	0.0000005
B	Parameter for pricing mortality intensity	1.14

$\gamma = 2$ and $\phi = 6$

Relative loss of wealth as a function of interest rate

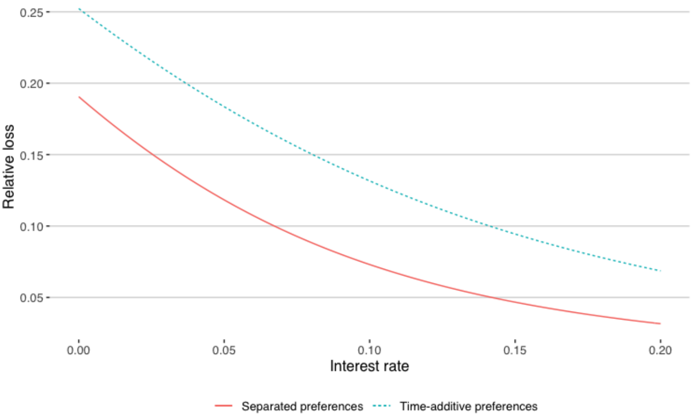

Figure: Interest rate

Relative loss of wealth as a function of market price of risk

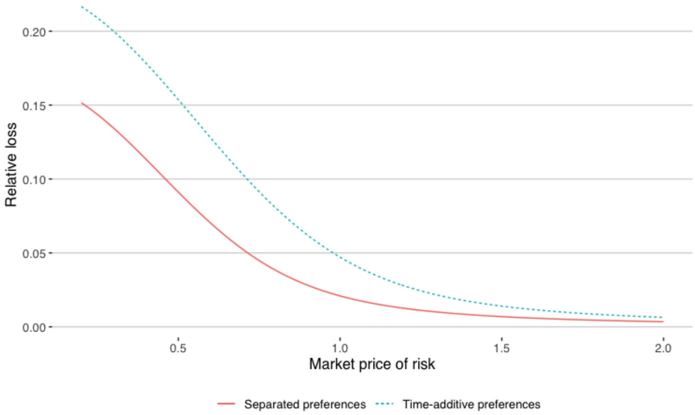

Figure: Market price of risk

Relative loss of wealth as a function of insurance pricing

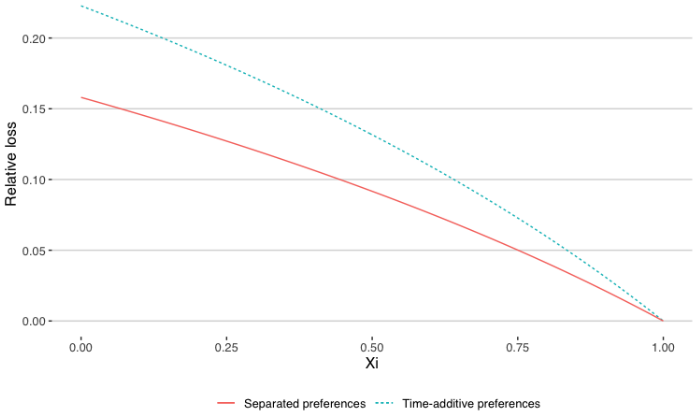

Figure: Insurance pricing

Relative loss of wealth as a function of lifetime unceartainty

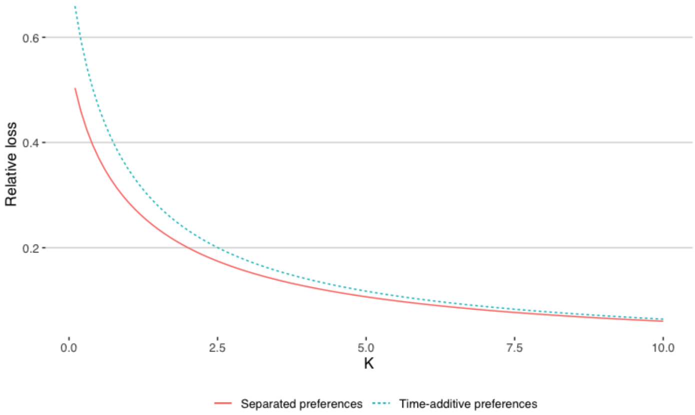

Figure: Lifetime unceartainty

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Questions?