Optimal Payoffs under smooth ambiguity

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joint with Steven Vanduffel and Morten Wilke

Optimal investment/payoff problem

- The field of optimal payoffs/investment problem has been extensively researched (Merton, 1969, 1971): $\mathbb{E}[U(X_T)]$
  - Risk preferences: Constant Relative Risk Aversion (CRRA) and Constant Absolute Risk Aversion (CARA)
  - Underlying asset dynamics: Geometric Brownian motions

Explanatory Slide: Payoff/Terminal Wealth vs. Investment Strategy

- Once X_T^* is determined, X_t^* for $t \in [0, T)$ can be derived using the pricing rule.
- There are two methods to express the wealth dynamics:
  - Using X_t^*, you can derive the wealth dynamics for dX_t^*.
  - Alternatively, directly express the wealth dynamics through the investment strategy.
- Compare coefficients between the two expressions of wealth dynamics to determine the investment strategy.

Abundant literature in the field

- Many different streams of extensions
  - developing further risk preferences, e.g. SAHARA utility by Chen et al. (2011)
  - maximize the option-type payoffs Carpenter (2000), Chen et al. (2019), Chen et al. (2024), e.g. $\mathbb{E}[U(\max(X_T - K, 0))]$
  - adding risk constraints to the optimization problem, e.g. Value-at-Risk, and Expected shortfall (Basak and Shapiro (2001), Chen et al. (2018a), Chen et al. (2018b))

What we do in our paper?

- This paper: given a static one-period financial market and an investor, we study the optimal payoff
  - KMM (Klibanoff-Marinacci-Mukerji-)Preferences, also called smooth ambiguity, (Klibanoff et al., 2005)
    - in our paper, we also consider *classical subjective expected utility theory* (CSEU) suggested by (Cerreia-Vioglio et al., 2013), special case of KMM
  - Payoff: all non-negative measurable functions of the risky asset's terminal value S_T; path-independent payoffs; non-linear payoffs allowed (c.f. e.g. Gollier (2011) for linear payoffs)

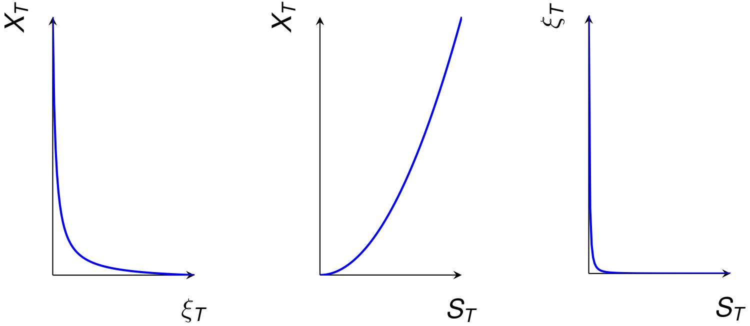
Why optimal payoffs in a static setting?

- If we allow continuous trading with zero transation costs, continuous trading shall be better.
- Optimal payoffs under smooth ambiguity in continuous-time setting (Bäuerle and Mahayni (2024))
  - drift uncertainty
  - power function of utility
  - power function describing ambiguity aversion
- We allow general utility function, general function describing the ambiguity aversion; both drift and volatility uncertainty

Main contributions

- First, we explicitly characterize and derive the optimal payoff for a CSEU and a KMM investor in our setting.
- Second, we show that a KMM investor (with second-order probabilities w and ambiguity attitude ϕ) opts for the same optimal payoff as a CSEU investor (c.f. equivalence result for linear payoffs in Taboga (2005) and Gollier (2011))
- Third, we show that optimal payoffs are *not* necessarily monotone in the stock price
    - providing a possible way to explain the pricing kernel puzzle

Explanatory slide: Optimal Payoff as Functions of ξ_T or S_T



Decreasing X_T of ξ_T Increasing X_T of S_T Decreasing ξ_T of S_T

- Empirical finding shows that the pricing kernel is not monotone in X_T or S_T $\rightarrow$ the pricing kernel puzzle (see e.g. Siddiqi and McMillan (2019))

No Ambiguity regarding $\mathbb{P}$

e.g. *Expected Utility*

$$\sup \mathbb{E}_{\mathbb{P}}[u(X_T)] \text{ s.t. } \mathbb{E}_{\mathbb{P}}[\xi_T X_T]$$

where ξ_T is monotone in S_T

$$\rightarrow X_T^* = I(\lambda \xi_T)$$

monotone in S_T

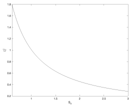


Figure: Pricing Kernel ξ_T as a function of S_T

Ambiguity regarding $\mathbb{P}$

e.g. *KMM utility*

$$\sup \mathbb{E}_{\mathbb{P}^{\tilde{w}}}[u(X_T)] \text{ s.t. } \mathbb{E}_{\mathbb{P}^{\tilde{w}}}[\xi_T^{\tilde{w}} X_T]$$

where $\xi_T^{\tilde{w}}$ might not be monotone in S_T

$$\rightarrow X_T^K(w, \phi) = I(\lambda \xi_T^{\tilde{w}})$$

might not be monotone in S_T

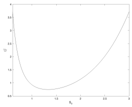


Figure: Pricing Kernel $\xi_T^{\tilde{w}}$ as a function of S_T

What comes next

- Optimal payoff under CSEU preference
- Optimal paoyff under KMM preferences
- Log-normal terminal asset prices

Financial Market

- Measurable space $(\Omega, \mathcal{F} = \sigma(S_T)), T > 0$
- **Payoffs:** initial budget $x_0 > 0$

$$\mathcal{X}(x_0) := \{X_T = g(S_T), g : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is } \mathcal{F}\text{-measurable}, \\ \mathbb{E}_{\mathbb{Q}}[e^{-rT} X_T] = x_0\Big\}$$

with pricing measure $\mathbb{Q}$

Investor – Risk and Uncertainty (CSEU)

First-order uncertainty (Risk)
What $\omega \in \Omega$ will materialize?

Second-order uncertainty (Ambiguity)
How likely is each $\mathbb{P}_i$?

Second-order uncertainty modelling with $\mathcal{P}$ and w

- Set of plausible probability measures

$$\mathcal{P} := \{\mathbb{P}_1, \ldots, \mathbb{P}_n\}, n \in \mathbb{N}.$$

- Second-order probabilities: Investor's confidence in each $\mathbb{P}_i$

$$\sum_{i=1}^{n} w_i = 1.$$

Risk attitude – Risk Aversion

The utility function $u : [0, \infty) \to \mathbb{R}$ satisfies the following properties:

- is strictly increasing ($u'(x) > 0$) and strictly concave ($u''(x) < 0$), and twice continously differentiable on $[0, \infty)$
- satisfies the Inada conditions, i. e.,

$$\lim_{x \to 0} u'(x) = \infty \text{ and } \lim_{x \to \infty} u'(x) = 0.$$

Classical-Subjective-Expected-Utility (Cerreia-Vioglio et al., 2013)

Given some payoff $X_T \in \mathcal{X}(x_0)$ the investor computes the CSEU utility

$$\mathcal{C}(X_T) = \sum_{i=1}^{n} w_i \cdot \mathbb{E}_{\mathbb{P}_i}[u(X_T)].$$

CSEU Problem

Given the initial budget $x_0 > 0$ the CSEU investor deems a payoff $X_T^{\mathcal{C}}(w) \in \mathcal{X}(x_0)$ **optimal** if it maximizes the CSEU utility, i.e.,

$$X_T^{\mathcal{C}}(w) = argmax_{X_T \in \mathcal{X}(x_0)} \mathcal{C}(X_T).$$

Optimal Payoff $X_T^{\mathcal{C}}$ under CSEU-Preferences

Proposition 1 (CSEU optimal payoff)

The optimal payoff under CSEU-preferences is given by

$$X_T^{\mathcal{C}}(w) = (u')^{-1}\left(\lambda \xi_T^w\right) := I\left(\lambda \xi_T^w\right)$$

where

1. $I(y) := (u')^{-1}(y), y > 0$ is the inverse of the marginal utility,
2. $\lambda > 0$ is chosen such that $X_T^{\mathcal{C}}(w) \in \mathcal{X}(x_0)$,
3. $\xi_T^w := \frac{e^{-rT}}{\sum_{i=1}^n w_i \cdot l_{\mathbb{P}_i}}$ is the subjective pricing kernel with $l_{\mathbb{P}_i} := \frac{\mathrm{d}\mathbb{P}_i}{\mathrm{d}\mathbb{Q}}$ as the likelihood ratio of $\mathbb{P}_i$ w.r.t. $\mathbb{Q}$.

Optimal Payoff $X_T^{\mathcal{C}}$ under CSEU-Preferences – Proof

Note that CSEU problem can then be rewritten as

$$\sup_{X_T \in \mathcal{X}(x_0)} \sum_{i=1}^{n} w_i \mathbb{E}_{\mathbb{P}_i}[u(X_T)] = \sup_{X_T \in \mathcal{X}(x_0)} \int_\Omega u(X_T(\omega)) \, \mathrm{d} \underbrace{\left(\sum_{i=1}^{n} w_i \mathbb{P}_i(\omega) \right)}_{=:\mathbb{P}^w}$$

$$= \sup_{X_T \in \mathcal{X}(x_0)} \mathbb{E}_{\mathbb{P}^w}[u(X_T)].$$

with budget constraint $x_0 = \mathbb{E}_{\mathbb{Q}}[e^{-rT} X_T] = \mathbb{E}_{\mathbb{P}^w}[\xi_T^w X_T]$ where

$$\xi_T^w := e^{-rT} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}^w} = \frac{e^{-rT}}{\sum_{i=1}^{n} w_i l_{\mathbb{P}_i}},$$

is the **subjective pricing kernel**. Use Cox and Huang (1989) to arrive at

$$X_T^{\mathcal{C}}(w) = I\left(\lambda \xi_T^w\right).$$

What comes next

- Optimal payoff under CSEU preference
- Optimal paoyff under KMM preferences
- Log-normal terminal asset prices

Investor – Risk and Ambiguity Attitude (under Smooth Ambiguity)

Risk Attitude – Risk Aversion

The utility function $u : [0, \infty) \to \mathbb{R}$

- is strictly increasing ($u'(x) > 0$) and strictly concave ($u''(x) < 0$), and twice continously differentiable on $[0, \infty)$
- satisfies the Inada conditions, i. e.,

$$\lim_{x \to 0} u'(x) = \infty \text{ and } \lim_{x \to \infty} u'(x) = 0.$$

Ambiguity Attitude – Ambiguity Aversion

The function ϕ describing ambiguity attitude is strictly increasing ($\phi'(U) > 0$), and strictly concave ($\phi''(U) < 0$) and twice continuously differentiable.

Investor – KMM Preferences (Klibanoff et al., 2005)

Given some payoff $X_T \in \mathcal{X}(x_0)$ the investor computes the KMM utility by

$$\mathcal{K}(X_T) = \sum_{i=1}^{n} w_i \cdot \phi(\mathbb{E}_{\mathbb{P}_i}[u(X_T)]).$$

KMM Problem

Given the initial budget $x_0 > 0$ the KMM investor deems a payoff $X_T^{\mathcal{K}}(w, \phi) \in \mathcal{X}(x_0)$ optimal if it maximizes the KMM utility, i.e.,

$$X_T^{\mathcal{K}}(w, \phi) = argmax_{X_T \in \mathcal{X}(x_0)} \mathcal{K}(X_T).$$

Characterization of Optimal Payoff $X_T^{\mathcal{K}}(w, \phi)$

A payoff $X_T^{\mathcal{K}}(w, \phi) \in \mathcal{X}(x_0)$ is KMM-optimal if and only if

$$\sum_{i=1}^{n} w_i \phi'(\mathbb{E}_{\mathbb{P}_i}[u(X_T^{\mathcal{K}}(w, \phi)]) \cdot \mathrm{d}_{X_T - X_T^{\mathcal{K}}(w,\phi)}(\mathrm{EU}_i)(X_T^{\mathcal{K}}(w, \phi)) \leq 0,$$

for all $X_T \in \mathcal{X}(x_0)$ where

$$\begin{aligned} &\mathrm{d}_{X_T - X_T^{\mathcal{K}}(w,\phi)}(\mathrm{EU}_i)(X_T^{\mathcal{K}}(w, \phi)) \\ &= \lim_{\epsilon \to 0} \frac{\mathbb{E}_{\mathbb{P}_i}[u(X_T^{\mathcal{K}}(w, \phi) + \epsilon(X_T - X_T^{\mathcal{K}}(w, \phi)))] - \mathbb{E}_{\mathbb{P}_i}[u(X_T^{\mathcal{K}}(w, \phi)))]}{\epsilon} \end{aligned}$$

denotes the Gateaux-differential of the functional

$$\mathrm{EU}_i : \mathcal{X} \to \mathbb{R}, X_T \to \mathbb{E}_{\mathbb{P}_i}[u(X_T))]$$

at $X_T^{\mathcal{K}}(w, \phi)$ in the direction of $X_T - X_T^{\mathcal{K}}(w, \phi)$.

Optimal Payoff $X_T^{\mathcal{K}}(w, \phi)$

Consider a KMM investor with utility function u, first-order probability measures in $\mathcal{P}$, second-order probabilities w, ambiguity attitude ϕ, and initial budget $x_0 > 0$. Assume that there exist second-order probabilities $\tilde{w}$ and w which solve

$$w_i = \kappa \cdot \frac{\tilde{w}_i}{\phi'(\mathbb{E}_{\mathbb{P}_i}[u(X_T^{\mathcal{C}}(\tilde{w}))])},$$

where $\kappa := \left(\sum_{j=1}^n \frac{\tilde{w}_j}{\phi'(\mathbb{E}_{\mathbb{P}_j}[u(X_T^{\mathcal{C}}(\tilde{w}))])} \right)^{-1}$ and $\sum_{i=1}^n w_i = 1$. Then a KMM-optimal payoff $X_T^{\mathcal{K}}(w, \phi)$ is given by

$$X_T^{\mathcal{K}}(w, \phi) = X_T^{\mathcal{C}}(\tilde{w}).$$

We call $\tilde{w}$ therefore CSEU-corresponding second-order probabilities.

Similar results in different settings by Gollier (2011), Guan et al. (2022) for linear payoffs

Monotonicity of the Optimal Payoff $X_T^{\mathcal{K}}(w,\phi)$ in S_T?

$$X_T^{\mathcal{K}}(w,\phi) = X_T^{\mathcal{C}}(\tilde{w}) = I\left(\lambda \frac{e^{-rT}}{\sum_{i=1}^{n} \tilde{w}_i l_{\mathbb{P}_i}}\right)$$

Remark 1 (Likelihood ratios if $\mathcal{F} = \sigma(S_T)$)

Let S_T have density $f^{\mathbb{P}_i} > 0$ and $f^{\mathbb{Q}} > 0$ under $\mathbb{P}_i$ and $\mathbb{Q}$, respectively. Then, if $\mathcal{F} = \sigma(S_T)$, we have for $i = 1, \ldots, n$ that

$$l_{\mathbb{P}_i} = \frac{\mathrm{d}\mathbb{P}_i}{\mathrm{d}\mathbb{Q}} = \frac{f^{\mathbb{P}_i}(S_T)}{f^{\mathbb{Q}}(S_T)}.$$

Monotonicity of the Optimal Payoff $X_T^{\mathcal{K}}(w, \phi)$ in S_T?

Proposition (Monotonicity of $X_T^{\mathcal{K}}(w, \phi)$)

If the subjective pricing kernel

$$\xi_T^{\tilde{w}} = e^{-rT} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}^{\tilde{w}}} = \frac{e^{-rT}}{\sum_{i=1} \tilde{w}_i l_{\mathbb{P}_i}}$$

is not monotone in S_T, then the optimal payoff

$$X_T^{\mathcal{K}}(w, \phi) = X_T^{\mathcal{C}}(\tilde{w}) = I(\lambda \xi_T^{\tilde{w}}),$$

is not monotone in S_T.

What comes next

- Optimal payoff under CSEU preference
- Optimal paoyff under KMM preferences
- Log-normal terminal asset prices

Lognormal Market Asset – Setup

Observe stock price with maturity T and volatility $\sigma_{\mathbb{Q}}$ of

$$S_T \overset{\mathbb{Q}}{\sim} \mathrm{LN}(rT - 1/2\sigma_{\mathbb{Q}}^2 T, \sigma_{\mathbb{Q}}^2 T)$$

Agent's first-order probability measures: $\mathcal{P} = \{\mathbb{P}_o, \mathbb{P}_p\}$ with

$$S_T \overset{\mathbb{P}_o}{\sim} \mathrm{LN}(\mu_o T - 1/2\sigma_o^2 T, \sigma_o^2 T)$$

$$S_T \overset{\mathbb{P}_p}{\sim} \mathrm{LN}(\mu_p T - 1/2\sigma_p^2 T, \sigma_p^2 T)$$

with $\mu_o > \mu_p > r$ and $\sigma_o \leq \sigma_p$.[1]

Agent's second-order probabilities: $w_p \in (0, 1)$ and $w_o = 1 - w_p$.

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[1] In principle, the following analysis can also be conducted for $\sigma_p < \sigma_o$.

Monotonicity of Subjective Pricing Kernel $\xi_T^{\tilde{w}}$

Let $i \in \{p, o\}$ and $\mu_i > r$ and $\sigma_p \geq \sigma_o$. Then the subjective pricing kernel

$$\xi_T^{\tilde{w}} = e^{-rT} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}^{\tilde{w}}}$$

for some second-order beliefs $\tilde{w} = (\tilde{w}_p, \tilde{w}_o)$ is

- strictly decreasing in S_T if $\sigma_o = \sigma_p = \sigma_{\mathbb{Q}}$,
- strictly decreasing in S_T for $S_T < \min(s_p^*, s_o^*)$ and strictly increasing in S_T for $S_T > \max(s_p^*, s_o^*)$ if $\sigma_o < \sigma_p < \sigma_{\mathbb{Q}}$,
- strictly increasing in S_T for $S_T < \min(s_p^*, s_o^*)$ and strictly decreasing in S_T for $S_T > \max(s_p^*, s_o^*)$ if $\sigma_{\mathbb{Q}} < \sigma_o < \sigma_p$.

where

$$s_i^* = \exp\left(\frac{r\sigma_i^2 - \mu_i\sigma_{\mathbb{Q}}^2}{\sigma_i^2 - \sigma_{\mathbb{Q}}^2} T\right), \quad \sigma_i \neq \sigma_{\mathbb{Q}}.$$

Numerical Example

- first-order beliefs: $\mathcal{P} = \{\mathbb{P}_p, \mathbb{P}_o\}$
- second-order beliefs: $w_p = 20\%, w_o = 80\%$
- risk attitude: $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma = 0.5$
- ambiguity attitude: $\phi(U) = -e^{-\eta U}$, $\eta = 1$
- risk-free interest rate: $r = 2\%$
- volatility of S_T under $\mathbb{Q}$: $\sigma_{\mathbb{Q}} = 20\%$

Change of weights $w \to \tilde{w}$

$$\mathcal{P} = \{\mathbb{P}_p, \mathbb{P}_o\}, \sigma_o = \sigma_p = \sigma_{\mathbb{Q}} = 20\%, w = (w_p, w_o) = (20\%, 80\%),$$
$$\phi(U) = -e^{-\eta U}, \eta > 0$$

η	$\tilde{w}_p$ (%)	$\tilde{w}_o$ (%)
1	20.77	79.23
10	28.32	71.68
100	87.67	12.33
∞	100.00	0.00

Table: CSEU-corresponding $\tilde{w}$ as a function of ambiguity aversion η

One-period model with Log-Normal Market Asset

No Ambiguity regarding $\mathbb{P}$

e.g. *Expected Utility*

$\sup \mathbb{E}_{\mathbb{P}}[u(X_T)]$ s.t. $\mathbb{E}_{\mathbb{P}}[\xi_T X_T]$

where ξ_T is monotone in S_T

$\rightarrow X_T^* = I(\lambda \xi_T)$
monotone in S_T

Ambiguity regarding $\mathbb{P}$

e.g. *KMM utility*

$\sup \mathbb{E}_{\mathbb{P}^{\tilde{w}}}[u(X_T)]$ s.t. $\mathbb{E}_{\mathbb{P}^{\tilde{w}}}[\xi_T^{\tilde{w}} X_T]$

where $\xi_T^{\tilde{w}}$ might not be monotone in S_T

$\rightarrow X_T^K(w, \phi) = I(\lambda \xi_T^{\tilde{w}})$
might not be monotone in S_T

Drift uncertainty

$$\mu_o - r = 5\% \text{ and } \mu_p - r = 3\%, \sigma_o = \sigma_p = \sigma_{\mathbb{Q}} = 20\%$$

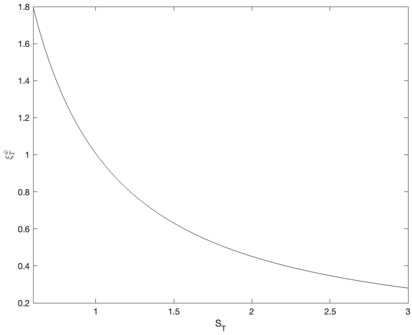

Figure: Subjective Pricing Kernel $\xi_T^{\tilde{w}}$

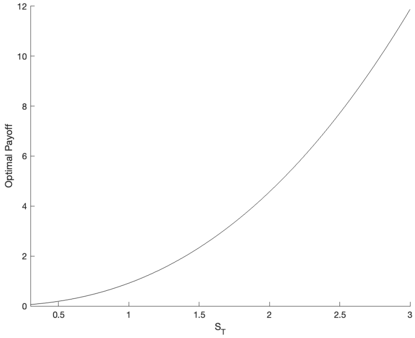

Figure: Optimal Payoff $X_T^{\mathcal{K}}(w, \phi)$

Volatility uncertainty

$\mu_o - r = 5\%$ and $\mu_p - r = 3\%$, $\sigma_o = 18\%, \sigma_p = 19\%, \sigma_{\mathbb{Q}} = 20\%$

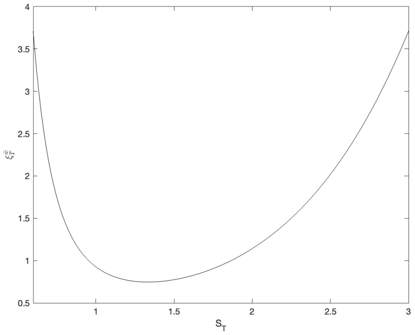

Figure: Subjective Pricing Kernel $\xi_T^{\tilde{w}}$

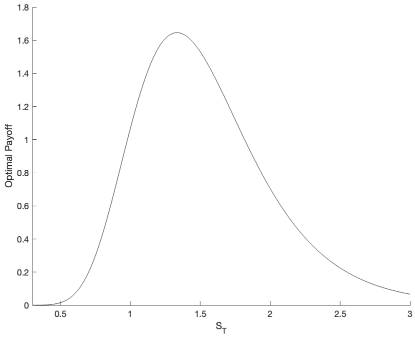

Figure: Optimal Payoff $X_T^{\mathcal{K}}(w, \phi)$

Conclusions

- Optimal payoff under CSEU preference
  - Explicit form
- Optimal paoyff under KMM preferences
  - Equivalent to CSEU solution
- Optimal payoff is not necessarily monotone in S_T
  - providing a possible way to explain the pricing kernel puzzle

For more details see

”Optimal Payoffs under **Smooth Ambiguity**” on EJOR

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Thank you very much for your attention!

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Proof

$X_T^{\mathcal{C}}(\tilde{w}) \in \mathcal{X}(x_0)$ satisfies

$$\sum_{i=1}^{n} \tilde{w}_i \cdot \mathrm{d}_{X_T - X_T^{\mathcal{C}}(\tilde{w})}(\mathrm{EU}_i)(X_T^{\mathcal{C}}(\tilde{w})) \leq 0, \tag{1}$$

$X_T^{\mathcal{K}}(w, \phi)$ satisfies

$$\sum_{i=1}^{n} w_i \phi'(\mathbb{E}_{\mathbb{P}_i}[u(X_T^{\mathcal{K}}(w, \phi))]) \cdot \mathrm{d}_{X_T - X_T^{\mathcal{K}}(w,\phi)}(\mathrm{EU}_i)(X_T^{\mathcal{K}}(w, \phi)) \leq 0. \tag{2}$$

Choose now

$$w_i = \kappa \cdot \frac{\tilde{w}_i}{\phi'(\mathbb{E}_{\mathbb{P}_i}[u(X_T^{\mathcal{C}}(\tilde{w}))])},$$

where $\kappa := \left(\sum_{j=1}^{n} \frac{\tilde{w}_j}{\phi'(\mathbb{E}_{\mathbb{P}_j}[u(X_T^{\mathcal{C}}(\tilde{w}))])} \right)^{-1}$ and $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0, 1]$ because $\phi'(\cdot) > 0$. Then inequality (2) is equivalent to

$$\sum_{i=1}^{n} \kappa \cdot \frac{\tilde{w}_i}{\phi'(\mathbb{E}_{\mathbb{P}_i}[u(X_T^{\mathcal{C}}(\tilde{w}))])} \cdot \phi'(\mathbb{E}_{\mathbb{P}_i}[u(X_T^{K}(w, \phi))]) \cdot \mathrm{d}_{X_T - X_T^{\mathcal{K}}(w,\phi)}(\mathrm{EU}_i)(X_T^{\mathcal{K}}(w, \phi)) \leq 0.$$

$\Rightarrow X_T^{K}(w, \phi) = X_T^{\mathcal{C}}(\tilde{w}).$