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Mean Variance Optimization for Participating Life Insurance Contracts

6^{th} Fudan-Ulm Symposium on Finance and Insurance
| Felix Fießinger & Mitja Stadje | 5^{th} September 2024

Motivation

- Investment problem: "$\max\limits_{\text{strategies}} \text{Mean}(Y) - \text{Variance}(Y)$"

$\rightarrow$ find optimal strategy

- Y: payoff of insurer

- Aim: find optimal terminal wealth and optimal investment strategy
- on top: show existence of all parameter

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Motivation

Participating Life Insurance Contracts:

- crucial role in the Life Sector

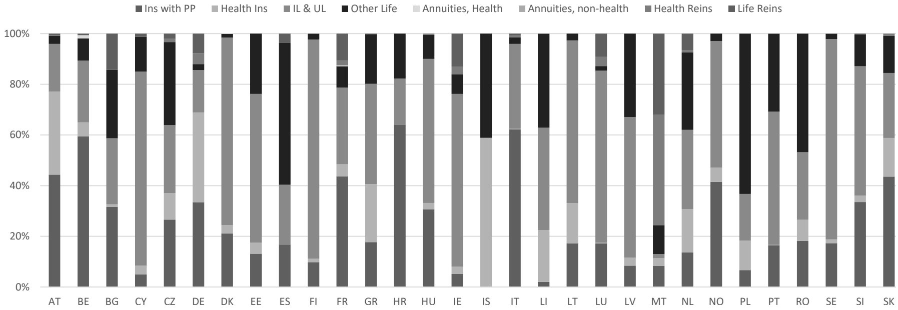

Figure: Market share in 2022 of the gross premium separated by the line of business in the life sector. Data source: European Insurance Overview from the EIOPA (2023)

Motivation

Participating Life Insurance Contracts:

- 2 main products: without or with guarantee for the policyholders

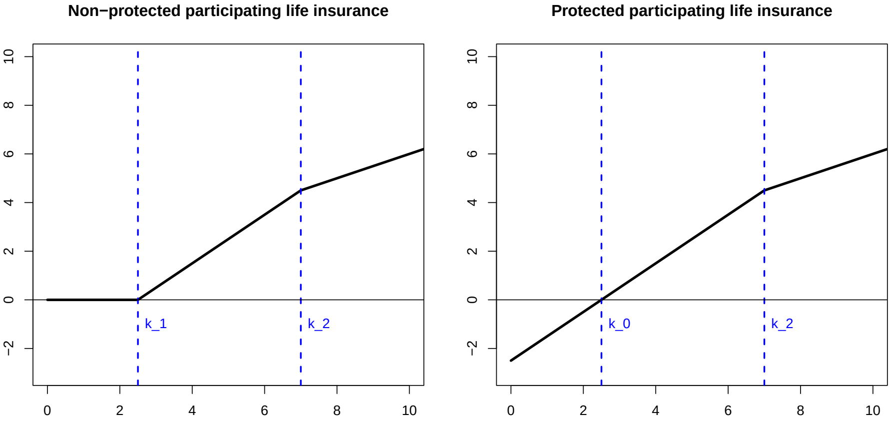

Figure: Payoffs of the insurer.

Motivation

- Main difficulty: non-linearity in $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\rightarrow$ give an equivalent problem as in Zhou & Li (2000)

- Lagrangian optimization, e.g., Basak & Shapiro (2001)

- optimize Participating Life Insurance Contracts:
- Problem: non-convexity resp. non-concavity of the payoff function
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Structure

Structure

d-dimensional Black-Scholes market

- $T > 0$ finite time horizon
- 1 risk-free asset: $\mathrm{d}B_t = B_t r_t \mathrm{d}t$
- d risky assets: $\mathrm{d}S_t^i = S_t^i \mu_t^i \mathrm{d}t + S_t^i \sigma_t^i \mathrm{d}W_t$
- strategies: u^i denotes the fraction of wealth invested in risky asset i, progressively measurable and square-integrable
- wealth: $\mathrm{d}X_t = X_t \left[r_t + u_t^T (\mu_t - r) \right] \mathrm{d}t + X_t u_t^T \sigma_t \mathrm{d}W_t$ with $X_0 = x_0$
- price density: $\mathrm{d}\xi_t = -\xi_t r_t \mathrm{d}t - \xi_t \kappa_t^T \mathrm{d}W_t$ with $\xi_0 = 1$ and the Sharpe ratio process $\kappa_t = (\sigma_t)^{-1}(\mu_t - r_t)$
- Interpretation: $\xi_T(\omega)$ Arrow-Debreu value per probability unit in state ω at time T
- Assume: μ and σ deterministic, σ bounded, bounded away from zero, invertible
- Assume: r, μ integrable, σ, κ square integrable over $[0, T]$

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Optimization Functional

- main target: $J(0,T,\hat{u},x_0) = \sup_{u \in \mathcal{U}} J(0,T,u,x_0)$
- functional J: $J(0,T,u,x_0) := \mathbb{E}[F(0,T,u,x_0)] - \gamma \mathrm{Var}(F(0,T,u,x_0))$ with risk aversion parameter $\gamma > 0$
- Function F:

$$F(s,t,u,x) := \alpha\left((X_t - k_1)_+ - k_0\right) - \alpha_2 (X_t - k_2)_+$$
$$= \begin{cases} -\alpha k_0 & X_t < k_1 \\ \alpha(X_t - k_1 - k_0) & k_1 \leq X_t < k_2 \\ \tilde{\alpha}(X_t - k_2) + \alpha(k_2 - k_1 - k_0) & X_t \geq k_2 \end{cases}$$

  where $X_s = x$, $0 \leq k_0, k_1 \leq k_2 < \infty$ with $k_0 + k_1 \leq k_2$, $0 \leq \alpha_2 < \alpha < \infty$ with $\tilde{\alpha} := \alpha - \alpha_2$ (Note: $0 < \tilde{\alpha} \leq \alpha$)
- Notation: $F(X_T)$ instead of $F(0,T,u,x)$ if u is clear

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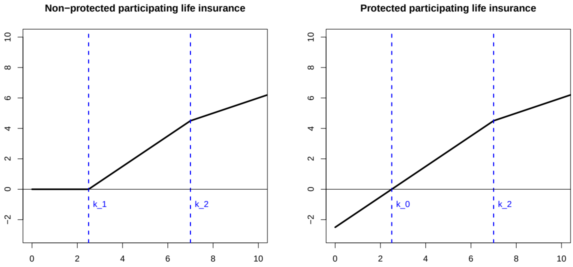
- non-protected contract: $\alpha = 1$, $k_0 = 0$, k_1 is the guarantee
- protected contract: $\alpha = 1$, k_0 is the guarantee, $k_1 = 0$



Structure

Equivalent Problem

- Problem: non-linearity in $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

- Solution:
  - value functional $\tilde{J}$: $\tilde{J}(0, T, u, x_0) := \mathbb{E}[\lambda F(0, T, u, x_0) - \gamma F(0, T, u, x_0)^2]$ with $\lambda = 1 + 2\gamma\mathbb{E}\left[F(0, T, \hat{u}, x_0)\right]$ where $\hat{u}$ is the optimal strategy
  - solve with λ as a parameter

Lemma

If $\hat{u}$ is an optimal strategy for J, it is also an optimal strategy for $\tilde{J}$.

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Optimal Terminal Wealth

Theorem

The optimal terminal wealth $\hat{X}_T$ is given by:

$$\hat{X}_T := \begin{cases} k_2 + \dfrac{\lambda\tilde{\alpha} - y\xi_T}{2\gamma\tilde{\alpha}^2} - \dfrac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0) & \xi_T \in (0, \xi_1^*] \\ k_2 & \xi_T \in (\tilde{\alpha}\hat{\xi}, \xi_2^*] \\ k_0 + k_1 + \dfrac{\lambda\alpha - y\xi_T}{2\gamma\alpha^2} & \xi_T \in (\alpha\hat{\xi}, \xi_3^*] \\ 0 & \textit{else} \end{cases},$$

where y is the Lagrangian multiplier which solves $\mathbb{E}[\xi_T\hat{X}_T(y)] = \xi_0 x_0$.

- $\hat{X}_T$ is the optimal wealth of the portfolio, i.e., before distributing
- insurer: $F(\hat{X}_T) = \alpha((\hat{X}_T - k_1)_+ - k_0) - \alpha_2(\hat{X}_T - k_2)_+$

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$$\hat{\xi} := \max\left\{0, \frac{\lambda - 2\gamma\alpha(k_2 - k_1 - k_0)}{y}\right\},$$

$$\bar{\xi} := \frac{\lambda\alpha}{y} + \frac{2\gamma\alpha^2 k_0}{y},$$

$$\tilde{\xi}_1^* := \tilde{\alpha}\hat{\xi} - \frac{2\gamma\tilde{\alpha}}{y}\left(\sqrt{\max\left\{0, (\alpha(k_0 + k_1) - \alpha_2 k_2)^2 - \alpha^2 k_0^2 + \frac{\lambda}{\gamma}(\alpha k_1 - \alpha_2 k_2)\right\}} - \tilde{\alpha} k_2\right),$$

$$\xi_1^* := \max\left\{0, \min\left\{\tilde{\alpha}\hat{\xi}, \tilde{\xi}_1^*\right\}\right\},$$

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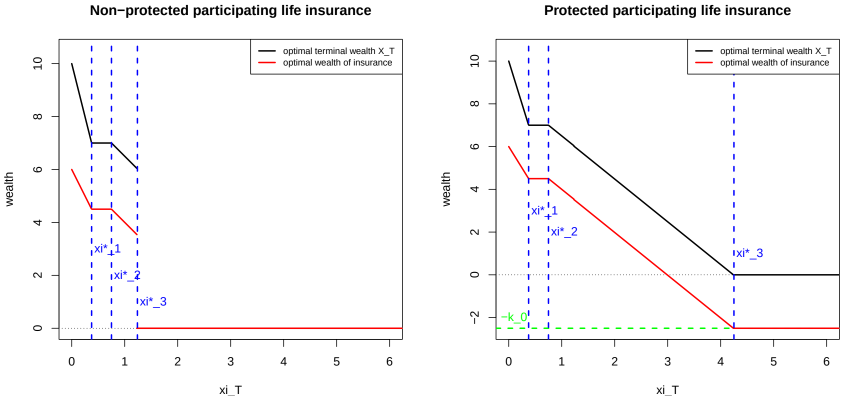
Optimal Terminal Wealth



Optimal Terminal Wealth

Theorem

In particular, the Lagrange multiplier exists. Note that we suppress the dependence on λ and y for the sake of simplicity in notation unless we state it otherwise in some proofs. Moreover, let ξ^* be as follows:

$$\xi^* := \begin{cases} \xi_3^* & ,\text{if } \xi_3^* > \alpha\hat{\xi} \\ \xi_2^* & ,\text{if } \xi_3^* = \alpha\hat{\xi}, \xi_2^* > \tilde{\alpha}\hat{\xi} \\ \xi_1^* & ,\text{if } \xi_3^* = \alpha\hat{\xi}, \xi_2^* = \tilde{\alpha}\hat{\xi} \end{cases}.$$

Then, it holds that $\xi^* > 0$ and $\hat{X}_T > 0$ for $\xi \in (0, \xi^*)$ and $\hat{X}_T = 0$ for $\xi > \xi^*$.

- $(0, \xi_1^*] \cup (\tilde{\alpha}\hat{\xi}, \xi_2^*] \cup (\alpha\hat{\xi}, \xi_3^*] = (0, \xi^*]$, i.e., these three intervals are connected
- $\hat{X}_T$ as a function of ξ is continuous and non-increasing in $(0, \xi^*) \cup (\xi^*, \infty)$
- There exists always a solution for λ and y and an equation system which can be numerically solved to determine them.

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Optimal Terminal Wealth of the Insurer

Corollary

The optimal payoff of the insurer is given by:

$$F(\hat{X}_T) = \begin{cases} \frac{\lambda\tilde{\alpha} - y\xi_T}{2\gamma\tilde{\alpha}} & \xi_T \in (0, \xi_1^*] \\ \alpha(k_2 - k_1 - k_0) & \xi_T \in (\tilde{\alpha}\hat{\xi}, \xi_2^*] \\ \frac{\lambda\alpha - y\xi_T}{2\gamma\alpha} & \xi_T \in (\alpha\hat{\xi}, \xi_3^*] \\ -\alpha k_0 & \textit{else} \end{cases},$$

where $\hat{\xi}$, ξ_1^*, ξ_2^*, and ξ_3^* are as before.

Optimal Strategy

Theorem

The optimal solution $\hat{u}$ is given by:

$$\hat{u}_t = (\sigma_t^T)^{-1} \kappa_t \frac{v_t}{\hat{X}_t},$$

Optimal Strategy

Theorem

$$
\begin{aligned}
v_t = & \left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)\right) \frac{e^{-\int_t^T r_s \mathrm{d}s}}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}} \varphi\left(d_1\left(\xi_1^*, t\right)\right) \\
& + \frac{y}{2\gamma\tilde{\alpha}^2} \xi_t e^{\int_t^T -(2r_s - \|\kappa_s\|^2)\mathrm{d}s} \left[\Phi\left(d_2\left(\xi_1^*, t\right)\right) - \frac{1}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}} \varphi\left(d_2\left(\xi_1^*, t\right)\right)\right] \\
& + k_2 \frac{e^{-\int_t^T r_s \mathrm{d}s}}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}} \left(\varphi\left(d_1\left(\xi_2^*, t\right)\right) - \varphi\left(d_1\left(\tilde{\alpha}\hat{\xi}, t\right)\right)\right) \\
& + \left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) \frac{e^{-\int_t^T r_s \mathrm{d}s}}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}} \left(\varphi\left(d_1\left(\xi_3^*, t\right)\right) - \varphi\left(d_1\left(\alpha\hat{\xi}, t\right)\right)\right) \\
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& \qquad \left. - \frac{1}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}} \left(\varphi\left(d_2\left(\xi_3^*, t\right)\right) - \varphi\left(d_2\left(\alpha\hat{\xi}, t\right)\right)\right)\right]
\end{aligned}
$$

Optimal Strategy

Theorem

$$
\begin{aligned}
\hat{X}_t = &\left(k_2 + \frac{\lambda}{2\gamma\tilde{\alpha}} - \frac{\alpha}{\tilde{\alpha}}(k_2 - k_1 - k_0)\right) e^{-\int_t^T r_s \mathrm{d}s} \Phi\left(d_1\left(\xi_1^*, t\right)\right) \\
&- \frac{y}{2\gamma\tilde{\alpha}^2} \xi_t e^{\int_t^T -(2r_s - \|\kappa_s\|^2)\mathrm{d}s} \Phi\left(d_2\left(\xi_1^*, t\right)\right) \\
&+ k_2 e^{-\int_t^T r_s \mathrm{d}s} \left(\Phi\left(d_1\left(\xi_2^*, t\right)\right) - \Phi\left(d_1\left(\tilde{\alpha}\hat{\xi}, t\right)\right)\right) \\
&+ \left(k_0 + k_1 + \frac{\lambda}{2\gamma\alpha}\right) e^{-\int_t^T r_s \mathrm{d}s} \left(\Phi\left(d_1\left(\xi_3^*, t\right)\right) - \Phi\left(d_1\left(\alpha\hat{\xi}, t\right)\right)\right) \\
&- \frac{y}{2\gamma\alpha^2} \xi_t e^{\int_t^T -(2r_s - \|\kappa_s\|^2)\mathrm{d}s} \left(\Phi\left(d_2\left(\xi_3^*, t\right)\right) - \Phi\left(d_2\left(\alpha\hat{\xi}, t\right)\right)\right),
\end{aligned}
$$

$$
d_1(x, t) = \frac{\ln x - \ln \xi_t + \int_t^T r_s - \frac{\|\kappa_s\|^2}{2} \mathrm{d}s}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}},
$$

$$
d_2(x, t) = \frac{\ln x - \ln \xi_t + \int_t^T r_s - \frac{3\|\kappa_s\|^2}{2} \mathrm{d}s}{\sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}} = d_1(x, t) - \sqrt{\int_t^T \|\kappa_s\|^2 \mathrm{d}s}
$$

Structure

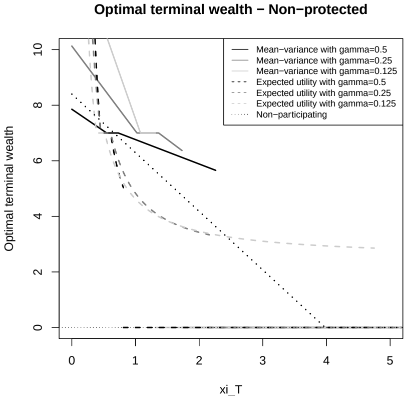
Parametrization

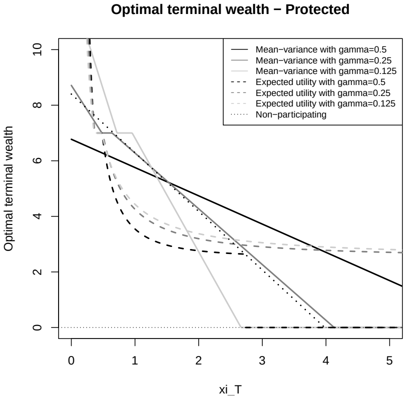
- $k_0 = 0$ / $k_0 = 2.5$
- $k_1 = 2.5$ / $k_1 = 0$
- $k_2 = 7$
- $\alpha = 1$
- $\alpha_2 = 0.25$
- $\gamma = 0.25$
- $x_0 = 4$
- $T = 10$
- $d = 1$
- $r = 0.02$
- $\mu = 0.08$
- $\sigma = 0.2$
- $\kappa = 0.3$
- $\delta = 0.01$
- $N = 1000$

For a comparison:

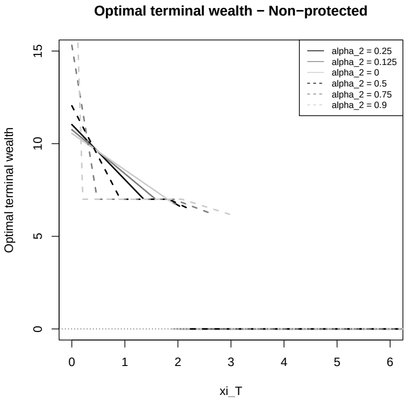
- S-shaped utility function: $U(x) = \begin{cases} x^{\tilde{\gamma}} & x \geq 0 \\ -\tilde{\lambda}(-x)^{-\tilde{\gamma}} & x < 0 \end{cases}$ with $\tilde{\lambda} = 2$ and different values for $\tilde{\gamma}$

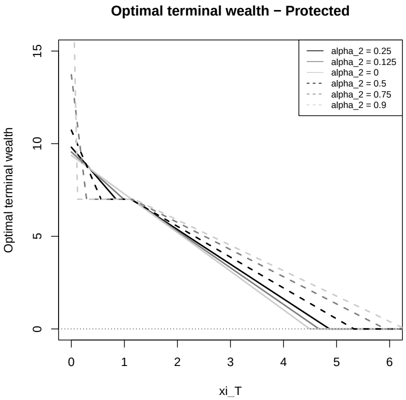
- comparison with results from expected utility of Lin *et al.* (2017)
- different risk aversion levels



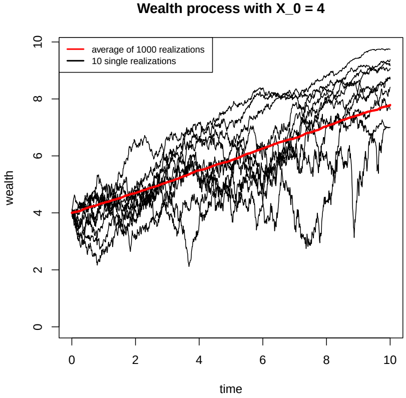


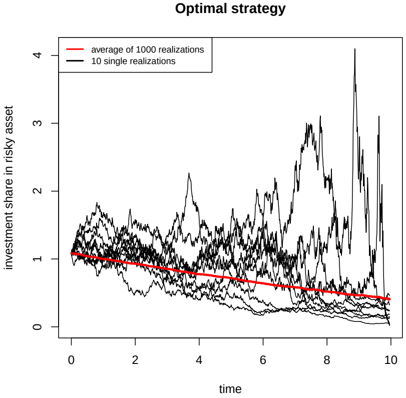
- influence of participation rate α_2





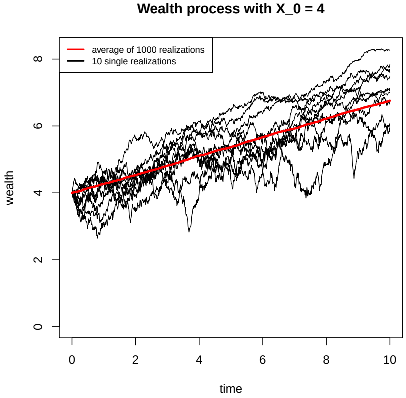
- Non-protected participating life insurance contract

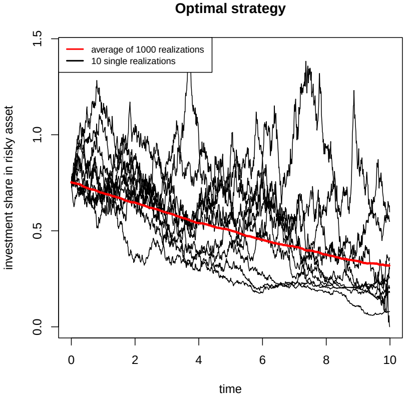




- comparably riskier strategies if economy evolves bad
- comparably safer strategies if economy evolves extremely good

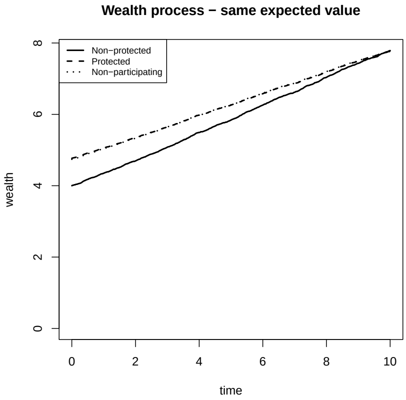
- Protected participating life insurance contract

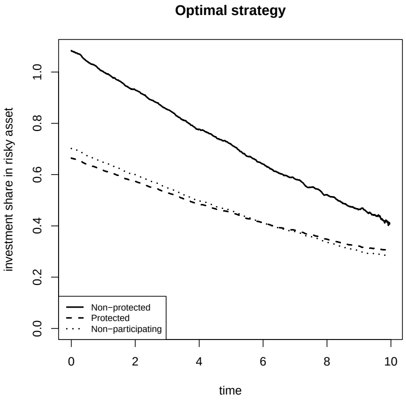




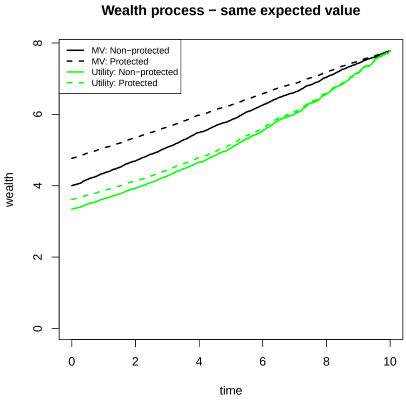
- extreme strategy changes if final value is close to k_2

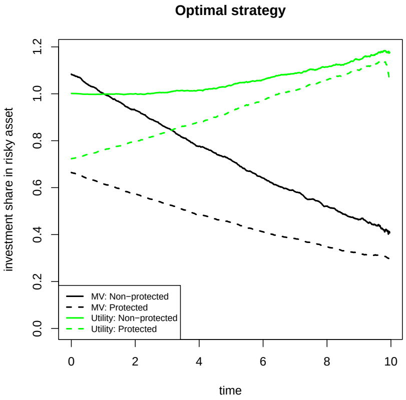
- Comparison between both participating contracts and a non-participation contract
- x_0 is chosen such that $\hat{X}_T$ is approximately equal in these cases





- Comparison between both participating contracts with mean-variance and expected utility
- x_0 is chosen such that $\hat{X}_T$ is approximately equal in these cases





Structure

Model Setup

Optimization in the Black-Scholes market

Numerical Results

Optimization in an Incomplete Market

Setting adaptations

- wealth process: $\mathrm{d}X_t = b(X_t, u_t)\mathrm{d}t + \sigma(X_t, u_t)\mathrm{d}W_t$ where b and σ are measurable functions which satisfy a uniform Lipschitz condition
- value functional: $V(t,x) = \sup_{u \in \mathcal{U}(t,x)} \tilde{J}(t, T, u, x) = \sup_{u \in \mathcal{U}(t,x)} \mathbb{E}[\lambda F(0, T, u, x_0) - \gamma F(0, T, u, x_0)^2]$ where $\mathcal{U}(t,x)$ denotes the subset of $\mathcal{U}$ with processes starting at t and $X_t = x$
- remember: $\lambda = 1 + 2\gamma \mathbb{E}\left[F(0, T, \hat{u}, x_0)\right]$

Theorem

If $V \in C^{1,2}$, then for every $\lambda \geq 0$ the optimal value functional V is the solution of the following SDE for all $(t,x) \in [0,T) \times \mathbb{R}$:

$$-\frac{\mathrm{d}V}{\mathrm{d}t}(t,x) - \sup_{u \in \mathcal{U}} \mathcal{L}^u V(t,x) = 0,$$

$$V(T,x) = F(T,T,0,x),$$

where the operator $\mathcal{L}^u$ is defined as

$$\mathcal{L}^u v(t,x) := b(x,u)\frac{\mathrm{d}V}{\mathrm{d}x}(t,x) + \frac{1}{2}\sigma(x,u)\sigma^T(x,u)\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}(t,x),$$

where tr denotes the trace of a matrix.

Theorem

Let the control space $\mathcal{U}$ be compact and the Hamiltonian H defined as usual, i.e.:

$$H : [0, T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

$$H(t, x, p, M) := \sup_{u \in \mathcal{U}} \left[b(x, u) \frac{\mathrm{d}V}{\mathrm{d}x}(t, x) + \frac{1}{2} \sigma(x, u) \sigma^T(x, u) \frac{\mathrm{d}^2 V}{\mathrm{d}x^2}(t, x) \right].$$

If V is locally bounded on $[0, T) \times \mathbb{R}$, then for every $\lambda \geq 0$ V is a viscosity solution of the following Hamilton-Jacobi-Bellman (HJB) equation for $(t, x) \in [0, T) \times \mathbb{R}$:

$$-\frac{\mathrm{d}V}{\mathrm{d}t}(t, x) - H\left(t, x, \frac{\mathrm{d}V}{\mathrm{d}x}(t, x), \frac{\mathrm{d}^2 V}{\mathrm{d}x^2}(t, x)\right) = 0,$$

$$V(T, x) = F(T, T, 0, x).$$

- can relax the assumption of $\mathcal{U}$ being compact

Thank you for your attention!

Preprint available: https://arxiv.org/pdf/2407.11761

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