Profit and loss decompositions

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Motivation 1: profit sharing

- **Profit sharing in life insurance**

  Mindestzuführungsverordnung (MindZV):

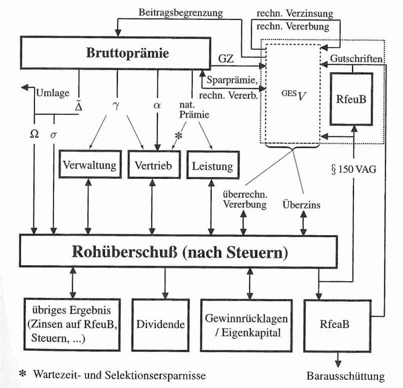
  Insurer has to refund at least

  - 90% of surplus coming from captial gains
  - 90% of surplus coming from insurance risk
  - 50% of the remaing surplus

  to the policyholders.

Motivation 1: profit sharing

- **Profit sharing in health insurance**

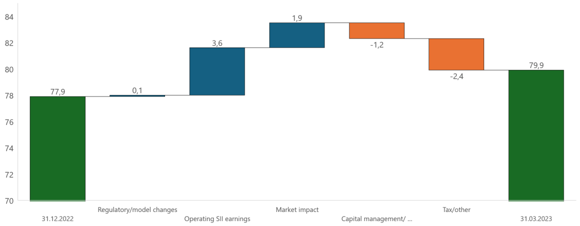


Milbrodt & Röhrs (2016, p. 405)

Motivation 2: explaining profits and losses

- **MCEV reporting**
  Analysis of earnings / movement analysis
- **IFRS 17 reporting**
  Movements in insurance contract liabilities analysed by components
- **Solvency II reporting**
  Analysis of change in SCR

Allianz – Group Financial Results 1Q 2023 – Own Funds (EUR bn)



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- **single time period**
- multiple time periods
- continuous time

Simple endowment insurance

Consider a 1-year endowment insurance that starts at age y.

Profit and loss at time 1

$$\underbrace{(1+i')}_{\text{zero coupon bond}} \cdot \underbrace{\frac{1-q_y}{1+i}}_{\text{premium}} - \underbrace{(1-\mathbf{1}_{\{T_y \le 1\}})}_{\text{survival benefit}}$$

$$= \underbrace{(1+i')}_{\text{compounding}} \cdot \left(\underbrace{\frac{1-q_y}{1+i} - \frac{1-q_y'}{1+i'}}_{\text{risk loading}} + \underbrace{\frac{1-q_y'}{1+i'} - \frac{1-\mathbf{1}_{\{T_y \le 1\}}}{1+i'}}_{\text{randomness}} \right)$$

$$= (1+i+x_3)^{x_4} \left(\frac{1-q_y}{1+i} - \frac{1-q_y-x_2}{1+i+x_3} - \frac{x_1}{1+i+x_3} \right)$$

Drivers of profits and losses

$x_1 = \mathbf{1}_{\{T_y \le 1\}} - q_y'$	unsystematic mortality risk
$x_2 = q_y' - q_y$	systematic mortality risk
$x_3 = i' - i$	interest rate risk
$x_4 = 1 - 0$	time

Endowment insurance

P&L function

$$f(x_1, x_2, x_3, x_4) := (1 + i + x_3)^{x_4} \left(\frac{1 - q_y}{1 + i} - \frac{1 - q_y - x_2}{1 + i + x_3} + \frac{x_1}{1 + i + x_3} \right)$$

P&L is zero for zero drivers

$$f(0, 0, 0, 0) = 0$$

Decomposition problem

Let $x = (x_1, \dots, x_d)$ be a vector of drivers and $f : \mathbb{R}^d \to \mathbb{R}$ a P&L function with $f(0, \dots, 0) = 0$. Decompose $f(x)$ into contributions

$$f(x) = g_1(x) + \cdots + g_d(x)$$

such that $g_k(x)$ is the P&L contribution of driver x_k.

Heuristic decomposition concepts

For simplicity, let $d = 2$ here.

Taylor approximation of first order

$$
\begin{aligned}
f(x_1, x_2) &= f(x_1, x_2) - f(0,0) \\
&= \underbrace{x_1\, \partial_{x_1} f(0,0)}_{=g_2(x_1,x_2)} + \underbrace{x_2\, \partial_{x_2} f(0,0)}_{=g_1(x_1,x_2)} + \mathit{Remainder}
\end{aligned}
$$

Properties

- may not exist
- not exact, i.e. $f \neq g_1 + g_2$

Heuristic decomposition concepts

For simplicity, let $d = 2$ here.

Sequential updating (SU) decomposition

$$\begin{aligned} f(x_1, x_2) &= f(x_1, x_2) - f(0,0) \\ &= \underbrace{f(x_1, x_2) - f(x_1, 0)}_{=g_2(x_1, x_2)} + \underbrace{f(x_1, 0) - f(0,0)}_{=g_1(x_1, x_2)} \end{aligned}$$

Alternative definition (different!)

$$f(x_1, x_2) = \underbrace{f(x_1, x_2) - f(0, x_2)}_{=g_1(x_1, x_2)} + \underbrace{f(0, x_2) - f(0,0)}_{=g_2(x_1, x_2)}$$

Properties

- always exists
- exact
- depends on order/labeling of risk factors

Heuristic decomposition concepts

For simplicity, let $d = 2$ here.

One at a time (OAT) decomposition

$$g_1(x_1, x_2) := f(x_1, 0) - f(0, 0)$$
$$g_2(x_1, x_2) := f(0, x_2) - f(0, 0)$$

Properties

- always exists
- not exact
- order invariant

Heuristic decomposition concepts

For simplicity, let $d = 2$ here.

Averaged sequential updating (ASU) decomposition

$$g_1(x_1, x_2) := \frac{f(x_1, 0) - f(0, 0)}{2} + \frac{f(x_1, x_2) - f(0, x_2)}{2}$$
$$g_2(x_1, x_2) := \frac{f(0, 0) - f(0, x_2)}{2} + \frac{f(x_1, x_2) - f(x_1, 0)}{2}$$

Properties

- always exists
- order invariant
- exact

Investment in a stock and a zero coupon bond

$$\text{P\&L} = \underbrace{(A(1) + B(1))}_{\text{`value at time 1'}} - \underbrace{(A(0) + B(0))}_{\text{`value at time 0'}}$$

Drivers of profits and losses

$$\begin{aligned} x_1 &= A(1) - A(0) && \text{change in stock value} \\ x_2 &= B(1) - B(0) && \text{change in bond value} \end{aligned}$$

P&L function

$$\begin{aligned} f(x_1, x_2) &= (A(0) + x_1) + (B(0) + x_2) - A(0) - B(0) \\ &= x_1 + x_2 \end{aligned}$$

It holds that $f(0,0) = 0$.

Investment in a stock and a zero coupon bond

Taylor decomposition

$$g_1(x) = x_1$$
$$g_2(x) = x_1$$

OAT decomposition

$$g_1(x) = x_1$$
$$g_2(x) = x_2$$

SU decomposition

$$g_1(x) = x_1$$
$$g_2(x) = x_2$$

alternative SU decomposition

$$g_1(x) = x_1$$
$$g_2(x) = x_2$$

ASU decomposition

$$g_1(x) = x_1$$
$$g_2(x) = x_2$$

Investment in a foreign fund

$$\text{P\&L} = \underbrace{A(1)R(1)}_{\text{`value at time 1'}} - \underbrace{A(0)R(0)}_{\text{`value at time 0'}}$$

Drivers of profits and losses

$$x_1 = A(1) - A(0) \qquad \text{change in fund value}$$
$$x_2 = R(1) - R(0) \qquad \text{change in currency value}$$

P&L function

$$\begin{aligned} f(x_1, x_2) &= (A(0) + x_1)(R(0) + x_2) - A(0)R(0) \\ &= x_1 R(0) + x_2 A(0) + x_1 x_2 \end{aligned}$$

It holds that $f(0, 0) = 0$.

Investment in a foreign fund

Taylor decomposition

$$g_1(x) = x_1 R(0)$$
$$g_2(x) = x_2 A(0)$$

OAT decomposition

$$g_1(x) = x_1 R(0)$$
$$g_2(x) = x_2 A(0)$$

SU decomposition

$$g_1(x) = x_1 R(0)$$
$$g_2(x) = x_2 A(0) + x_1 x_2$$

alternative SU decomposition

$$g_1(x) = x_1 R(0) + x_1 x_2$$
$$g_2(x) = x_2 A(0)$$

ASU decomposition

$$g_1(x) = x_1 R(0) + \frac{1}{2} x_1 x_2$$
$$g_2(x) = x_2 A(0) + \frac{1}{2} x_1 x_2$$

Endowment insurance

P&L function

$$f(x) := (1+i+x_3)^{x_4}\left(\frac{1-q_y}{1+i} - \frac{1-q_y-x_2-x_1}{1+i+x_3}\right)$$

SU decomposition (1,2,3,4)

$$g_1(x) = x_1$$
$$g_2(x) = x_2\frac{1}{1+i}$$
$$g_3(x) = \frac{1-q_y-x_2}{1+i} - \frac{1-q_y-x_2}{1+i+x_3}$$
$$g_4(x) = x_4\left(\frac{1-q_y}{1+i} - \frac{1-q_y-x_2+x_1}{1+i+x_3}\right)$$

SU decompositions (3,2,1,4)

$$g_1(x) = x_1\frac{1}{1+i+x_3}$$
$$g_2(x) = x_2\frac{1}{1+i+x_3}$$
$$g_3(x) = \frac{1-q_y}{1+i} - \frac{1-q_y}{1+i+x_3}$$
$$g_4(x) = x_4\left(\frac{1-q_y}{1+i} - \frac{1-q_y-x_2+x_1}{1+i+x_3}\right)$$

SU decomposition (4,3,2,1)

$$g_1(x) = x_1$$
$$g_2(x) = x_2$$
$$g_3(x) = x_3\frac{1-q_y}{1+i}$$
$$g_4(x) = 0$$

....

Endowment insurance

	SU (4,3,2,1) and (4,3,1,2)	contribution formula
unsystematic mortality risk	$\left(\mathbf{1}_{\{T_y \le 1\}} - q'_y\right) V_1$	
systematic mortality risk	$\left(q'_y - q_y\right) V_1$	$\left(q'_y - q_y\right) V_1$
interest rate risk	$(i' - i)(V_0 + p)$	$(i' - i)(V_0 + P_0)$
time	$(1 + i')S_0$	

$$V_t := \text{prospective reserve at time } t$$
$$S_t := \text{surplus at time } t$$
$$P_0 := \text{premium at time zero}$$

Intermediate summary

- **SU decompostion** exact, not order invariant.
- **OAT decomposition** is not exact, order invariant.
- **ASU decomposition** is exact, order invariant

The classical contribution formula is a **SU decomposition** with the update order

1. time
2. interest rate risk
3. unsystematic & systematic mortality risk

Example: insurance reporting

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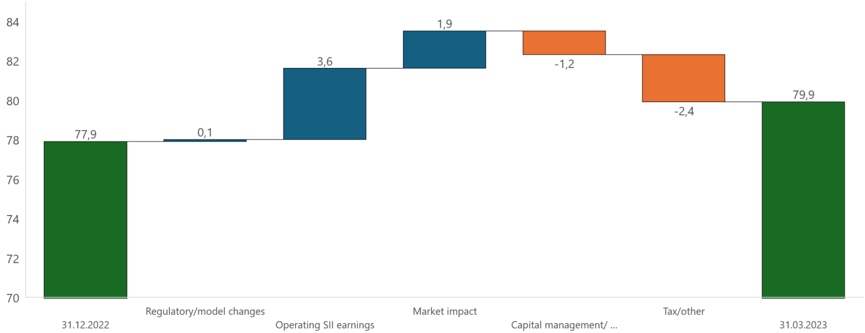

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Axiomatic decomposition concept

Cooperative game theory: Let

$$f : \{0,1\}^d \to \mathbb{R}, \quad f(0,\dots,0) = 0$$

be the P&L of a game with $d \in \mathbb{N}$ players.

Theorem (Shapley, 1953)

There exists a unique decomposition with the following properties:

- **Exactness**: $f = g_1 + \cdots + g_d$
- **Order invariance**: permutation $\pi : \{1,\dots d\} \to \{1,\dots d\}$
  $f'(x) = f(x_{\pi(1)},\dots,x_{\pi(d)}) \implies g'_{\pi(k)}(x) = g_k(x_{\pi(1)},\dots,x_{\pi(d)})$
- **Additivity**: $f'' = f + f' \implies g''_k = g_k + g'_k,\ k \in \{1,\dots,d\}$
- **Dummy neutrality**: $f(x_1,\dots,x_d)$ constant in $x_k \implies g_k = 0$

This unqiue decomposition is the ASU decomposition.

Axiomatic decomposition concept

Shapley-Shubik construction: For any given

$$f : \mathbb{R}^d \to \mathbb{R}, \quad f(0, \dots, 0) = 0$$

we define the family of games

$$f^x : \{0,1\}^d \to \mathbb{R}, \quad f^x(z) := f(x_1 z_1, \dots, x_d z_d), \qquad x \in \mathbb{R}.$$

(We indeed have $f^x(0, \dots, 0) = 0$.)

Lemma

- It holds that $f(x) = f^x(1, \dots, 1)$, $x \in \mathbb{R}$.
- Let $(g_1^x, \dots, g_d^x)$, $x \in \mathbb{R}^d$, be the ASU decompositions of the games f^x, $x \in \mathbb{R}^d$. Then
$$g_k(x) := g_k^x(1, \dots, 1), \quad x \in \mathbb{R},$$
is the ASU decomposition of f.

Axiomatic decomposition concept

Popular belief: *"The Shapley axioms uniquely characterize the ASU decomposition."* Wrong!

The ASU decomposition of a mapping $f : \mathbb{R}^d \to \mathbb{R}$ with $f(0, \ldots, 0) = 0$ is **not** uniquely characterized by the following properties:

- **Exactness**: $f = g_1 + \cdots + g_d$
- **Order invariance**: permutation $\pi : \{1, \ldots d\} \to \{1, \ldots d\}$
  $f'(x) = f(x_{\pi(1)}, \ldots, x_{\pi(d)}) \implies g'_{\pi(k)}(x) = g_k(x_{\pi(1)}, \ldots, x_{\pi(d)})$
- **Additivity**: $f'' = f + f' \implies g''_k = g_k + g'_k, k \in \{1, \ldots, d\}$
- **Dummy neutrality**: $f(x_1, \ldots, x_d)$ constant in $x_k \implies g_k = 0$

Axiomatic decomposition concept

Unique characterization of ASU

- Shapley (1953): $f : \{0,1\}^d \to \mathbb{R}$, $f(0,\dots,0) = 0$ **(4 axioms)**
- Sprumont (1998), Friedman & Moulin (1999): $f : [0,\infty)^d \to \mathbb{R}$, $f(0,\dots,0) = 0$, monotone & differentiable **(5/7 axioms)**
- Christiansen & Junike (2024): $f : \mathbb{R}^d \to \mathbb{R}$, $f(0,\dots,0) = 0$, Borel measurable **(8 axioms)**

Theorem

The ASU decomposition is uniquely characterized by the following properties:

1. **Exactness**
2. **Order invariance**
3. **Dummy neutrality**
4. **Linearity (not just Additivity)**: $\alpha, \beta \in \mathbb{R}$ real numbers
   $f'' = \alpha f + \beta f' \implies g'' = \alpha g + \beta g'$
5. **Monotonicity**: $x_k \mapsto f(x)$ monotone $\implies x_k \mapsto g_k(x)$ monotone
6. **Sampling consistency**: $x^n \to x$ real sequence
   $\lim_{n\to\infty} f(x^n) = f(x)$ and $\lim_{n\to\infty} g_k(x^n)$ exists $\forall k \implies \lim_{n\to\infty} g_k(x^n) = g_k(x) \forall k$
7. **Approximation consistency**: $\lim_{n\to\infty} f^n = f$ and $\lim_{n\to\infty} g_k^n$ exists $\forall k \implies \lim_{n\to\infty} g_k^n = g_k \forall k$
8. **Unit invariance**: $A = diag(\alpha_1, \dots, \alpha_d)$ diagonal matrix with $\alpha_1, \dots \alpha_d \in \mathbb{R} \setminus \{0\}$
   $f'(x) = f(Ax)$ for all $x \in \mathbb{R}^d \implies g_k'(x) = g_k(Ax)$ for all $x \in \mathbb{R}^d$ and $k \in \{1, \dots, d\}$

Axiomatic decomposition concept

Unique characterization of ASU

- Shapley (1953): $f : \{0,1\}^d \to \mathbb{R}$, $f(0,\dots,0) = 0$ **(4 axioms)**
- Sprumont (1998), Friedman & Moulin (1999): $f : [0,\infty)^d \to \mathbb{R}$, $f(0,\dots,0) = 0$, monotone & differentiable **(5/7 axioms)**
- Christiansen & Junike (2024): $f : \mathbb{R}^d \to \mathbb{R}$, $f(0,\dots,0) = 0$, Borel measurable **(8 axioms)**

Theorem

The ASU decomposition is uniquely characterized by the following properties:

1. **Exactness**
2. **Order invariance**
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4. **Linearity (not just Additivity)**: $\alpha, \beta \in \mathbb{R}$ real numbers
   $f'' = \alpha f + \beta f' \implies g'' = \alpha g + \beta g'$
5. **Monotonicity**: $x_k \mapsto f(x)$ monotone $\implies$ $x_k \mapsto g_k(x)$ monotone
6. **Sampling consistency**: $x^n \to x$ real sequence
   $\lim_{n\to\infty} f(x^n) = f(x)$ and $\lim_{n\to\infty} g_k(x^n)$ exists $\forall k \implies \lim_{n\to\infty} g_k(x^n) = g_k(x) \forall k$
7. **Approximation consistency**: $\lim_{n\to\infty} f^n = f$ and $\lim_{n\to\infty} g_k^n$ exists $\forall k \implies \lim_{n\to\infty} g_k^n = g_k \forall k$
8. **Unit invariance**: $A = diag(\alpha_1, \dots, \alpha_d)$ diagonal matrix with $\alpha_1, \dots \alpha_d \in \mathbb{R} \setminus \{0\}$
   $f'(x) = f(Ax)$ for all $x \in \mathbb{R}^d \implies g_k'(x) = g_k(Ax)$ for all $x \in \mathbb{R}^d$ and $k \in \{1, \dots, d\}$

Intermediate summary

- **ASU decomposition** is the unique decomposition that is exact, order invariant, dummy neutral, linear, monotone, sampling consistent, approximation consistent, unit invariant.

- single time period ✓
- **multiple time periods**
- continuous time

Endowment insurance

Consider an n-year endowment insurance that starts at age y.

Profit and loss at time m

$$S_t = \underbrace{\prod_{k=1}^{t}(1+i'_k)}_{\text{compounding}} \left(\underbrace{\frac{{}_n p_y}{(1+i)^n}}_{\text{premium}} - \underbrace{\frac{\mathbf{1}_{\{T_y>t\}}}{\prod_{k=1}^{t}(1+i'_k)} \frac{{}_{n-t}p_{y+t}}{(1+i)^{n-t}}}_{\text{expected survival benefit}} \right)$$

Drivers of profits and losses as trajectories $x_1, \dots, x_d : \{0, 1, \dots, n\} \to \mathbb{R}$

$x_1(t) - x_1(t-1) = \mathbf{1}_{\{T_y>t-1\}}\left(\mathbf{1}_{\{T_y \le t\}} - q'_{y+t-1}\right)$	unsystematic mortality risk
$x_2(t) - x_2(t-1) = \mathbf{1}_{\{T_y>t-1\}}\left(q'_{y+t-1} - q_{y+t-1}\right)$	systematic mortality risk
$x_3(t) - x_3(t-1) = i'_t - i$	interest rate risk
$x_4(t) - x_4(t-1) = t - (t-1)$	time

for $t \geq 1$ and $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$.

Endowment insurance

Available information at time t given by t-stopping

$$x_k^t := x_k(\cdot \wedge t), \quad k \in \{1, \ldots, d\}$$

It holds that $x_k^0 = \mathbf{0}$ for all k, where $\mathbf{0}$ denotes the zero function.

The **profit and loss** at time $t \in \{0, \ldots, n\}$ can be represented as

$$S_t = f(x_1^t, \ldots, x_d^t)$$

$$= \prod_{k=1}^{t}(1 + i_k')\left(\frac{{}_n p_y}{(1+i)^n} - \frac{\mathbf{1}_{\{T_y > t\}}}{\prod_{k=1}^{t}(1+i_k')}\frac{\prod_{k=t}^{n-1}(1 - q_{y+k})}{(1+i)^{n-t}}\right)$$

for the **P&L functional** $f : (\mathbb{R}^{\{0,\ldots,n\}})^d \to \mathbb{R}$ defined by

$$f(x_1, x_2, x_3, x_4)$$

$$= \prod_{k=1}^{x_4(n)}(1 + i + \Delta x_3(k))\left(\frac{{}_n p_y}{(1+i)^n} - \prod_{k=1}^{n}\frac{1 - q_{y+k-1} - \Delta x_1(k) - \Delta x_2(k)}{1 + i + \Delta x_3(k)}\right)$$

It holds that $f(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = 0$,

Decomposition problem

Let $\mathcal{D} := \{\text{functions } h : \mathbb{N}_0 \to \mathbb{R} \text{ with } h(0) = 0\}$.

Decomposition problem

For given

- time-dynamic drivers $x_1, \ldots, x_d \in \mathcal{D}$
- a P&L functional $f : \mathcal{D}^d \to \mathbb{R}$ with $f(\mathbf{0}, \ldots, \mathbf{0}) = 0$

find P&L contribution functionals $g_1, \ldots, g_d : \mathcal{D}^d \to \mathbb{R}$ such that

$$f(x_1^t, \ldots, x_d^t) = g_1(x_1^t, \ldots, x_d^t) + \cdots + g_d(x_1^t, \ldots, x_d^t) \qquad \forall t \in \mathbb{N}_0$$

Heuristic decomposition concepts

For simplicity let $d = 2$.

Sequential updating (SU) decomposition

$$
\begin{aligned}
f(x_1^t, x_2^t) &= f(x_1^t, x_2^t) - f(\mathbf{0}, \mathbf{0}) \\
&= \sum_{k=1}^{\infty} \Big(f\big((x_1^t)^k), (x_2^t)^k\big) - f\big((x_1^t)^{k-1}, (x_2^t)^{k-1}\big) \Big) \\
&= \underbrace{\sum_{k=1}^{\infty} \Big(f\big((x_1^t)^k), (x_2^t)^k\big) - f\big((x_1^t)^k, (x_2^t)^{k-1}\big)}_{=:g_2(x_1^t, x_2^t)} \\
&\quad + \underbrace{\sum_{k=1}^{\infty} \Big(f\big((x_1^t)^k), (x_2^t)^{k-1}\big) - f\big((x_1^t)^{k-1}, (x_2^t)^{k-1}\big)}_{=:g_1(x_1^t, x_2^t)}
\end{aligned}
$$

Properties

- exact
- order dependent

Heuristic solutions

For simplicity let $d = 2$.

One at a time (OAT) decomposition

$$g_1(x_1^t, x_2^t) = \sum_{k=1}^{\infty} \Big(f\big((x_1^t)^k), (x_2^t)^k\big) - f\big((x_1^t)^{k-1}, (x_2^t)^{k-1}\big) \Big)$$

$$g_2(x_1^t, x_2^t) = \sum_{k=1}^{\infty} \Big(f\big((x_1^t)^{k-1}), (x_2^t)^k\big) - f\big((x_1^t)^{k-1}, (x_2^t)^{k-1}\big) \Big)$$

Properties

- not exact
- order invariant

Heuristic solutions

Averaged sequential updating (ASU) decomposition

... arithmetic average of all variants of the SU decompostions ...

Properties

- order invariant
- exact
- ...

Endowment insurance

e.g. SU decomposition with update order (4,3,1,2) or (4,3,2,1)

unsystematic mortality risk on $(t-1,t]$	$\mathbf{1}_{\{T_y>t-1\}}\left(\mathbf{1}_{\{T_y\leq t\}}-q'_{y+t-1}\right)V_t$
systematic mortality risk on $(t-1,t]$	$\mathbf{1}_{\{T_y>t-1\}}\left(q'_{y+t-1}-q_{y+t-1}\right)V_t$
interest rate risk on $(t-1,t]$	$\mathbf{1}_{\{T_y>t-1\}}(i'_t-i)(V_{t-1}+P_{t-1})$
time value of money on $(t-1,t]$	$(1+i'_t)S_{t-1}$

$V_t :=$ prospective reserve at time t

$S_t :=$ surplus at time t

$P_t :=$ premium at time t

Endowment insurance

e.g. SU decomposition with update order (4,3,1,2) or (4,3,2,1)

unsystematic mortality risk on $(t-1,t]$	$\mathbf{1}_{\{T_y>t-1\}}\left(\mathbf{1}_{\{T_y\leq t\}}-q'_{y+t-1}\right)V_t$
systematic mortality risk on $(t-1,t]$	$\mathbf{1}_{\{T_y>t-1\}}\left(q'_{y+t-1}-q_{y+t-1}\right)V_t$
interest rate risk on $(t-1,t]$	$\mathbf{1}_{\{T_y>t-1\}}(i'_t-i)(V_{t-1}+P_{t-1})$
time value of money on $(t-1,t]$	$(1+i'_t)S_{t-1}$

$V_t :=$ prospective reserve at time t

$S_t :=$ surplus at time t

$P_t :=$ premium at time t

classical **contribution formula**, cf. Milbrodt & Helbig (1999, p. 541)

Intermediate summary

- SU / OAT / ASU decompositions have **straightforward extensions to multiple periods**

- The classical contribution formula is an **SU decomposition** (order dependent) with the update order
  1. time
  2. interest rate risk
  3. unsystematic and systematic mortality risk

  The **ASU decomposition** would be order invariant.

- single time period ✓
- multiple time periods ✓
- **continuous time**

Decomposition problem

Let $\mathcal{D} := \{\text{càdlàg functions } h : [0,\infty) \to \mathbb{R} \text{ with } h(0) = 0\}$.

Decomposition problem

For given

- time-dynamic drivers $x_1, \ldots, x_d \in \mathcal{D}$
- a P&L functional $f : \mathcal{D}^d \to \mathbb{R}$ with $f(\mathbf{0}, \ldots, \mathbf{0}) = 0$

find P&L contribution functionals $g_1, \ldots, g_d : \mathcal{D}^d \to \mathbb{R}$ such that

$$f(x_1^t, \ldots, x_d^t) = g_1(x_1^t, \ldots, x_d^t) + \cdots + g_d(x_1^t, \ldots, x_d^t).$$

Available information at time t given by t-stopping

$$x_k^t := x_k(\cdot \wedge t), \quad k \in \{1, \ldots, d\}$$

Note that $x_k^0 = \mathbf{0}$, so that

$$f(x_1^0, \ldots, x_d^0) = f(\mathbf{0}, \ldots, \mathbf{0}) = 0.$$

Heuristic decomposition concepts

Let $\mathcal{T} = \{0 = t_0 < t_1 < \cdots\}$ be a time grid with $t_n \to \infty$.

SU decomposition with respect to $\mathcal{T}$

$$g_1(x_1, x_2) = \sum_{k=1}^{\infty} \Big(f(x_1^{t_k}, x_2^{t_{k-1}}) - f(x_1^{t_{k-1}}, x_2^{t_{k-1}}) \Big)$$

$$g_2(x_1, x_2) = \sum_{k=1}^{\infty} \Big(f(x_1^{t_k}, x_2^{t_k}) - f(x_1^{t_k}, x_2^{t_{k-1}}) \Big)$$

- exact
- order dependent
- time grid dependent

Heuristic decomposition concepts

Let $\mathcal{T} = \{0 = t_0 < t_1 < \cdots\}$ be a time grid with $t_n \to \infty$.

OAT decomposition with respect to $\mathcal{T}$

$$g_1(x_1, x_2) = \sum_{k=1}^{\infty} \Big(f(x_1^{t_k}, x_2^{t_{k-1}}) - f(x_1^{t_{k-1}}, x_2^{t_{k-1}}) \Big)$$

$$g_2(x_1, x_2) = \sum_{k=1}^{\infty} \Big(f(x_1^{t_{k-1}}, x_2^{t_k}) - f(x_1^{t_{k-1}}, x_2^{t_{k-1}}) \Big)$$

- not exact
- order invariant
- time grid dependent

Heuristic decomposition concepts

Let $\mathcal{T} = \{0 = t_0 < t_1 < \cdots\}$ be a time grid with $t_n \to \infty$.

ASU decomposition with respect to $\mathcal{T}$

... arithmetic average of all variants of the SU decompostions ...

- order invariant
- exact
- time grid dependent

Heuristic decomposition concepts

Infinitesimal sequential updating (ISU) decomposition

$$ISU = \lim_{|\mathcal{T}| \to 0} SU^{\mathcal{T}}$$

Infinitesimal one at a time updating (IOAT) decomposition

$$IOAT = \lim_{|\mathcal{T}| \to 0} OAT^{\mathcal{T}}$$

Infinitesimal average sequential updating (IASU) decomposition

$$IASU = \lim_{|\mathcal{T}| \to 0} ASU^{\mathcal{T}}$$

- time grid invariant provided that the limits exist
- the other properties still hold

Example

Consider the case

$$f(x_1^t, x_2^t) = x_1(t)x_2(t), \quad t \in [0, 1].$$

- For $\mathcal{T}_1 = \mathbb{N}_0$ we get the **ASU decomposition**

$$g_1(x_1^1, x_2^1) = \frac{x_1(1)x_2(1)}{2}$$
$$g_2(x_1^1, x_2^1) = \frac{x_1(1)x_2(1)}{2}$$

- For $\mathcal{T}_2 = \{k/2 : k \in \mathbb{N}_0\}$ we get the **ASU decomposition**

$$g_1(x_1^1, x_2^1) = \frac{x_1(0.5)x_2(0.5)}{2} + (x_1(1) - x_1(0.5))\frac{x_2(0.5) + x_2(1)}{2}$$
$$g_2(x_1^1, x_2^1) = \frac{x_1(0.5)x_2(0.5)}{2} + \frac{x_1(0.5) + x_1(1)}{2}(x_2(1) - x_2(0.5))$$

The ASU decomposition depends on the choice of the grid.

Example

- For $\mathcal{T}_n = \{k/n : k \in \mathbb{N}_0\}$ we get the **ASU decomposition**

$$g_1^n(x_1^1, x_2^1) = \sum_{k=1}^{n} (x_1(k/n) - x_1((k-1)/n)) \frac{x_2((k-1)/n) + x_2(k/n)}{2}$$

$$g_2^n(x_1^1, x_2^1) = \sum_{k=1}^{n} \frac{x_1((k-1)/n) + x_1(k/n)}{2} (x_2(k/n) - x_2((k-1)/n))$$

- If x_1, x_2 are differentiable, then we get the **IASU decomposition**

$$\begin{aligned} f(x_1^1, x_2^1) &= x_1(1) - x_2(1) \\ &= \lim_{n\to\infty} g_1^n(x_1^1, x_2^1) + \lim_{n\to\infty} g_2^n(x_1^1, x_2^1) \\ &= \int_0^1 x_2(t)\mathrm{d}x_1(t) + \int_0^1 x_1(t)\mathrm{d}x_2(t) \end{aligned}$$

exact, order invariant, time grid invariant

Endowment insurance

Consider an n-year endowment insurance that starts at age y.

Profit and loss at time m

$$S_t = \underbrace{e^{\int_0^t \phi'(u)\mathrm{d}u}}_{\text{compounding}} \left(\underbrace{\frac{e^{-\int_0^t \mu(y+u)\mathrm{d}u}}{(1+i)^n}}_{\text{premium}} - \underbrace{\frac{\mathbf{1}_{\{T_y>t\}}}{e^{\int_0^t \phi'(u)\mathrm{d}u}} \frac{e^{-\int_0^t \mu(y+u)\mathrm{d}u}}{(1+i)^{n-t}}}_{\text{expected survival benefit}} \right)$$

Drivers of profits and losses as trajectories $x_1, \ldots, x_d : \{0, 1, \ldots, n\} \to \mathbb{R}$

$dx_1(t) = dN(t) - \mathbf{1}_{\{T_y \geq t\}} \mu(y+t)dt$	unsystematic mortality risk
$dx_2(t) = \mathbf{1}_{\{T_y \geq t\}} \big(\mu'(y+t) - \mu(y+t)\big) dt$	systematic mortality risk
$dx_3(t) = \big(\phi'(t) - \ln(1+i)\big) dt$	interest rate risk
$dx_4(t) = dt$	time

for $t > 0$ and $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$.

Endowment insurance

Available information at time t given by t-stopping

$$x_k^t := x_k(\cdot \wedge t), \quad k \in \{1, \ldots, d\}$$

It holds that $x_k^0 = \mathbf{0}$ for all k, where $\mathbf{0}$ denotes the zero function.

The **profit and loss** at time $t \in \{0, \ldots, n\}$ can be represented as

$$\begin{aligned} S_t &= f(x_1^t, \ldots, x_d^t) \\ &= e^{\int_0^t \phi'(u)\mathrm{d}u} \left(\frac{e^{-\int_0^t \mu(y+u)\mathrm{d}u}}{(1+i)^n} - \frac{\mathbf{1}_{\{T_y > t\}}}{e^{\int_0^t \phi'(u)\mathrm{d}u}} \frac{e^{-\int_0^t \mu(y+u)\mathrm{d}u}}{(1+i)^{n-t}} \right) \end{aligned}$$

for the **P&L functional** $f : (\mathbb{R}^{\{0,\ldots,n\}})^d \to \mathbb{R}$ defined by

$$f(x_1, x_2, x_3, x_4) = e^{\int_0^{x_4(n)} x_3(u)\mathrm{d}u} \left(\frac{{}_n p_y}{(1+i)^n} - \frac{e^{-\int_0^n (\mu(y+u)+x_1(u)+x_2(u))du}}{e^{\int_0^n (\phi(u)+x_3(u))du}} \right)$$

It holds that $f(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = 0$.

Endowment insurance

SU decomposition for **any update order**

unsystematic mortality risk at t	$V_t\big(dN(t) - \mathbf{1}_{\{T_y \geq t\}}\mu'(y+t)dt\big)$
systematic mortality risk at t	$V_t\,\mathbf{1}_{\{T_y \geq t\}}\big(\mu'(y+t) - \mu(y+t)\big)dt$
interest rate risk at t	$V_t\,\mathbf{1}_{\{T_y \geq t\}}\big(\phi'(t) - \ln(1+i)\big)dt$
time value of money at t	$S_t\,\phi'(t)dt$

$V_t :=$ prospective reserve at time t

$S_t :=$ surplus at time t

Conjecture: *"The IASU decomposition is order invariant."* **Wrong!**

Let x_1 and x_2 be paths of two Brownian motions and

$$f(x_1^t, x_2^t) = x_1(t)x_2(t).$$

ISU decomposition with update order (1,2)

$$g_1(x_1^t, x_2^t) = \int_0^t x_2(u)dx_1(u)$$
$$g_2(x_1^t, x_2^t) = \int_0^t x_1(u)dx_2(u) + [x_1, x_2](t)$$

IASU decomposition

$$g_1(x_1^t, x_2^t) = \int_0^t x_2(u)dx_1(u) + \frac{1}{2}[x_1, x_2](t)$$
$$g_2(x_1^t, x_2^t) = \int_0^t x_1(u)dx_2(u) + \frac{1}{2}[x_1, x_2](t)$$

Axiomatic decomposition concept

Theorem (Christiansen & Junike, 2025)

For a “large class” of P&L functionals, the **IASU decomposition** is the only decomposition with the following properties:

1. **Exactness**
2. **Order invariance**
3. **Dummy neutrality**
4. **Linearity**
5. **Monotonicity**
6. **Sampling consistency**
7. **Approximation consistency**
8. **Unit invariance**
9. **Non-anticipativeness**

Investment in a foreign fund

$$\text{P\&L} = \underbrace{A(t)R(t)}_{\text{`value at time t'}} - \underbrace{A(0)R(0)}_{\text{`value at time 0'}}$$

Drivers of profits and losses

$$x_1(t) = A(t) - A(0) \qquad \text{change in fund value}$$
$$x_2(t) = R(t) - R(0) \qquad \text{change in currency value}$$

P&L functional

$$\begin{aligned} f(x_1^t, x_2^t) &= (A(0) + x_1(t))(R(0) + x_2(t)) - A(0)R(0) \\ &= x_1(t)R(0) + x_2(t)A(0) + x_1(t)x_2(t) \end{aligned}$$

It holds that $f(\mathbf{0}, \mathbf{0}) = 0$.

IASU decomposition for semimartingales A and R

$$\begin{aligned} f(x_1^t, x_2^t) = &\left(x_1(t)R(0) + \int_0^t x_2(u)dx_1(u) + \frac{1}{2}[x_1, x_2](t) \right) \\ &+ \left(x_2(t)A(0) + \int_0^t x_2(u)dx_1(u) + \frac{1}{2}[x_1, x_2](t) \right) \end{aligned}$$

Intermediate summary

- ISU / IOAT / IASU decompositions extend discrete-time concepts to continuous time and are time grid invariant
- **ISU decomposition** often order invariant if the drivers have zero covariation, but in general order dependent

- single time period ✓
- multiple time periods ✓
- continuous time ✓

- **curse of dimensionality**

Computational effort

- **(I)SU decomposition**: $\mathcal{O}(d)$
- **(I)ASU decomposition**: $\mathcal{O}(2^{d-1}d)$

d	$2^{d-1}d$
1	1
2	2
3	6
4	16
5	96
6	224
7	512
⋮	⋮
20	10.485.760

2SU Approximation

....